Preface

This text book is the Grade 10 High School Mathematics written in accordance with the new Curriculum. This volume is a continuation of the first one which is already prescribed for Grade 10 students. Most of the chapters in this text spirally develop the mathematical concept and skill, and try to instil in students a deeper and thorough understanding of basic ideas in mathematics.

The general aims of teaching mathematics at high school level are specified as follows: At the end of Secondary Level, students should be able

- ➢ to gain basic mathematical knowledge and understanding
- ➢ to acquire necessary mathematical skill
- ➢ to apply mathematical knowledge and skill in real-life situations, and
- ➢ to interest in, and appreciation of mathematics, together with the development of maths-related values.

Chapter 1 introduces basic ideas of coordinate geometry. This chapter expresses the properties of mid-point, properties of slopes, distance between two points and equation of a line.

Chapter 2 emphasizes the basic properties of exponents, radicals and solving the exponential equations. The main purpose is to use the properties of the exponents and radicals correctly.

Chapter 3 emphasizes the basic properties of logarithms, change of base and using common logarithms. Ideas mainly developed in this chapter are the algebra of logarithm of a positive number.

Chapter 4 is concerned with the functions. The study of functions aim to develop an understanding of functions as mappings and to recognize functions as relations between sets. Also the basic properties of composite functions and inverse functions are included.

Chapter 5 deals with the methods for solving a quadratic equation are discussed. Also discrimination of quadratic and the graphical solution of quadratic inequations are expressed.

Chapter 6 introduces basic ideas in absolute value of linear function and inequality of absolute.

Chapter 7 is concerned with the probability of an event. The uses of tree diagrams and table of outcomes are emphasized in calculating the probability. Method of calculating expected frequency is involved.

Chapter 8 and 9 attempt to stress the formal structure of geometry and integrate geometry with arithmetic and algebra. Emphasis is laid down upon the use of precise language in the statements of definitions, postulate, and theorems. Chapter 8 is about basis ideas of similar triangles, angle bisector theorem, and extension of the Pythagoras Theorem.

Chapter 9 deals with the definitions of circle, and properties of chords.

Chapter 10 which is the last chapter of this text, attempts to broaden the student's understanding of geometric properties, and interrelations between sides and angles of plane figures. A general feature of an angle is first presented followed by relation between degree and radian measures. Then, the definitions of six trigonometric ratios are extended to include all angles.

After learning this course, students will develop and practise higher order thinking skills: comprehension, analysis, synthesis and evaluation. They will be able to participate actively in all lessons through the **5 C's** as important **21^{st} century skills for learning**:

- **Collaboration** - in lesson students will be working in groups, to share ideas with their classmates and to find the solution together
- **Communication** - students will develop verbal and non-verbal communication skills in group works
- **Critical thinking and problem solving** - students will be given interesting problems to solve-finding and explaining solutions, looking for correcting errors
- **Creativity and innovation** - thinking 'outside the box' is an important 21st century skill. Students will be encouraged to explore new ideas and solve problems in new ways.
- **Citizenship** - students will join the school community and develop fairness and conflict resolution skills.

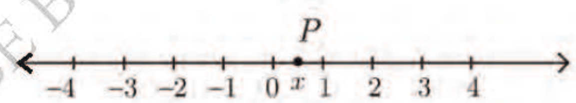
An important feature of these texts for high school level is that ideas, concepts, principles and methods are integrated within each branch of mathematics and across the branches.

Finally, students are encouraged to work through the text, to acquire basic mathematical knowledge and skill, to apply them to real-life situation and to develop mathematical thinking and reasoning.

Chapter 1

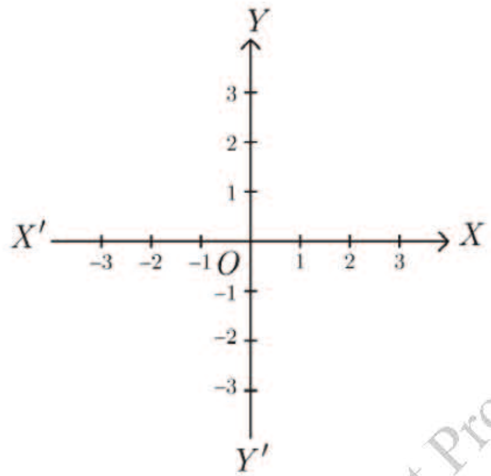
Introduction to Coordinate Geometry

René Descarts' greatest contribution to mathematics was the discovery of coordinate systems and their applications to problems of geometry. Coordinate system used in this book is referred to as Cartesian coordinate system.
We have seen how coordinate systems work on a line.

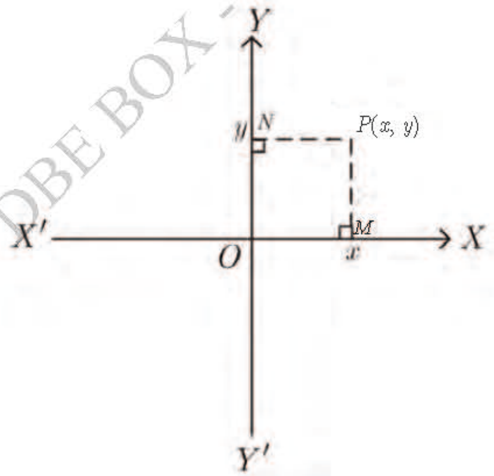


Once we have set up a coordinate system on a line, every number corresponds to a point on the line and every point on the line corresponds to a number. We shall now extend this idea to the points in a plane, a point will correspond not to a single number, but to an ordered pair of numbers. The scheme works like this.

First we take two perpendicular lines, a horizontal line $X'OX$(the $\boldsymbol{X}$-**axis**) and a vertical line $Y'OY$(the $\boldsymbol{Y}$-**axis**), they intersect at zero point in the $\boldsymbol{XY}$-**plane**. The zero point which is the intersection of these two lines is called **the origin**, normally labeled O. On the X-axis, values to the right are positive and those to the left are negative. On the Y-axis, values above the origin are positive and those below are negative.

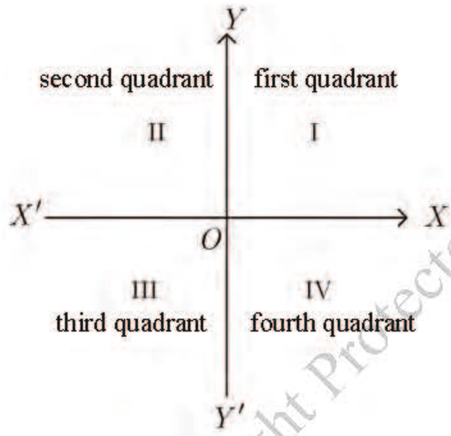


We can now describe any point P of the plane by an ordered pair of numbers, as follows. Draw PM and PN, perpendicular to the X-axis and Y-axis. Let x be the coordinate of M on the line $X'OX$ and y be the coordinate of N on the line $Y'OY$.

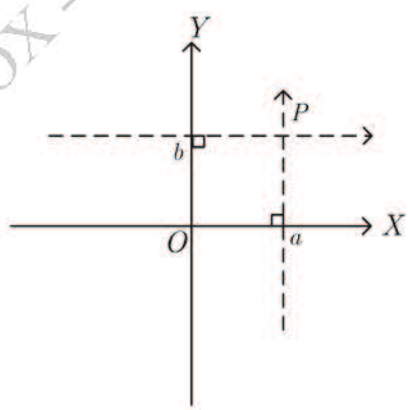


The number x and y are called the x-coordinate and y-coordinate of P respectively. In short, we indicate that P has these coordinates by writing $P(x, y)$. In particular, the origin O has coordinates $(0, 0)$.

Just as single line separates the plane into two parts (each of which is a half-plane), so the two axes separate the XY-plane into four parts, called quadrants. The four quadrants are identified by the Roman numerals, I, II, III, IV.



We have shown that under the scheme, every point P determines an ordered pair of real numbers. Does it work in reverse? That is, does every ordered pair (a, b) of real numbers determine a point? It is easy to see that the answer is "Yes".



Mark a point on the X-axis, so that the x-coordinate of that point is a. Draw a perpendicular line passes through that point. Then draw another perpendicular line at the point in which the y-coordinate is b. The point where these perpendiculars intersect is the point with coordinates (a, b).

Thus we have a one-to-one correspondence between the points of the plane and the ordered pairs of real numbers. Such a correspondence is called a **rectangular coordinate system**.

To describe such a coordinate system, we need to choose
(i) a line $X'OX$ to be the X-axis,
(ii) a line $Y'OY$ perpendicular to $X'OX$ to be the Y-axis, and
(iii) a positive direction on each of the axes.

Once we have made these choices, the coordinate systems on both axes are determined, and they in turn determine the coordinates of all points of the plane. This plane is referred as XY-plane. With reference to a coordinate system, every point P determines an ordered pair (a, b) and every ordered pair (a, b) determines a point.

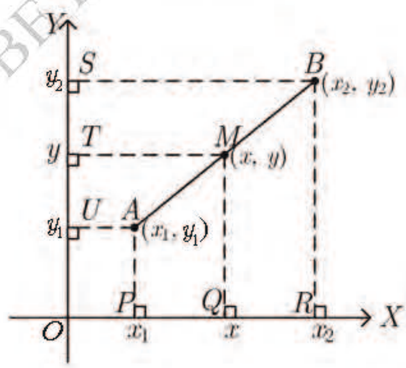
1.1 Midpoint and Length of a Line Segment

Every line segment has a midpoint. Both the midpoint and length of a line segment can be found by using the coordinates of the endpoints.

Midpoint of a Line Segment in XY-plane

The midpoint M of a line segment is the halfway point between the two endpoints.

To find the coordinates of the midpoint of a non-horizontal, non-vertical line segment joining the two given points in the XY-plane.



Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two given points and $M(x, y)$ be the midpoint of AB. We will now find the coordinates of M in terms of x_1, x_2, y_1 and y_2.

Draw the perpendicular lines AP, MQ, BR to the X-axis as shown. Draw the perpendicular lines AU, MT, BS to Y-axis. Clearly AU, MT, BS are horizontal lines while AP, MQ, BR are vertical lines. The points M and Q

have the same x-coordinates and M and T have the same y-coordinates.

Q is the midpoint of PR, the x-coordinate of Q is $\dfrac{x_1 + x_2}{2}$.

T is the midpoint of SU, the y-coordinate of T is $\dfrac{y_1 + y_2}{2}$.

$\therefore$ x-coordinate of midpoint $M = x = \dfrac{x_1 + x_2}{2}$, and

y-coordinate of midpoint $M = y = \dfrac{y_1 + y_2}{2}$.

Midpoint Formula:

The coordinates of midpoint of $A(x_1, y_1)$ and $B(x_2, y_2)$ are

$$M(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Example 1.

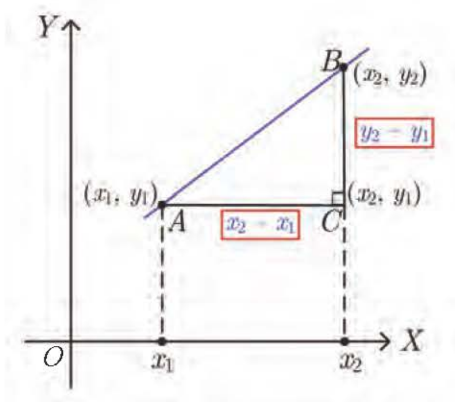
Find the coordinates of the midpoint of PQ with endpoints $P(2, 5)$ and $Q(6, 1)$.

Solution

If M is the midpoint of PQ, then

$$M = \left(\frac{2+6}{2}, \frac{5+1}{2}\right) = (4, 3).$$

The Length of a Line Segment in XY-plane



Let AB be a non-horizontal, non-vertical line segment with $A(x_1, y_1)$ and $B(x_2, y_2)$. Complete a right-angled triangle ABC as shown.
The line segment is the hypotenuse of a right-angled triangle. You may use Pythagoras' theorem to find the length of a straight line segment.

Horizontal distance $AC = x_2 - x_1$

Vertical distance $BC = y_2 - y_1$

By Pythagoras' theorem:

$AB^2 = AC^2 + BC^2$

$AB = \sqrt{AC^2 + BC^2}$

Therefore, the length of line segment AB is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Distance Formula:

The distance between two points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Example 2.

Find the length of PQ with endpoints $P(1, 5)$ and $Q(4, 1)$.

Solution

Let $(1, 5) = (x_1, y_1)$ and $(4, 1) = (x_2, y_2)$.

$$\begin{aligned}\text{Length of } PQ &= \sqrt{(4-1)^2 + (1-5)^2} \\ &= \sqrt{(3)^2 + (-4)^2} \\ &= \sqrt{9+16} = \sqrt{25} = 5.\end{aligned}$$

Example 3.

A is the point $(5, -3)$ and B is the point $(-2, 1)$.

(a) Find the midpoint of AB. (b) Find the length of AB.

Solution

(a) Let $(5, -3) = (x_1, y_1)$ and $(-2, 1) = (x_2, y_2)$.

$$\text{Midpoint of } AB = \left(\frac{5+(-2)}{2}, \frac{-3+1}{2}\right) = \left(\frac{3}{2}, \frac{-2}{2}\right) = (1.5, -1)$$

(b)

$$\text{Length of } AB = \sqrt{(-2-5)^2 + (1-(-3))^2} = \sqrt{49+16} = \sqrt{65}$$

Example 4.

The point $M(a,4)$ is the midpoint of the line segment with endpoints at $A(1,3)$ and $B(5,b)$. Find the value of a and of b.

Solution

Let (x_1,y_1) be $(1,3)$ and (x_2,y_2) be $(5,b)$.

$$\begin{aligned}\text{Midpoint of } AB &= \left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)\\ (a,4) &= \left(\frac{1+5}{2},\frac{3+b}{2}\right).\end{aligned}$$

Equating the x-coordinates and y-coordinates respectively, we get

$$a=\frac{1+5}{2}\quad,\quad 4=\frac{3+b}{2}$$
$$a=3\quad,\quad b=5.$$

Example 5.

The distance between two points $R(9,a)$ and $S(a+1,2)$ is 6. Find the two possible values of a.

Solution

Let (x_1,y_1) be $(9,a)$ and (x_2,y_2) be $(a+1,2)$.

Using distance formula $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ and $RS=6$, we get

$$\begin{aligned}\sqrt{(a+1-9)^2+(2-a)^2} &= 6\\ (a-8)^2+(2-a)^2 &= 6^2\\ a^2-16a+64+4-4a+a^2 &= 36\\ 2a^2-20a+32 &= 0\\ a^2-10a+16 &= 0\\ (a-8)(a-2) &= 0\\ a-8=0 &\text{ or } a-2=0\\ a=8 &\text{ or } a=2.\end{aligned}$$

Example 6.
In a parallelogram $ABCD$, three vertices are $A(-3,1), B(2,4)$ and $C(3,1)$.
(a) Find the midpoint of the diagonal AC. (b) Find the coordinates of D.

Solution

(a) Midpoint of $AC = \left(\frac{-3+3}{2}, \frac{1+1}{2}\right) = (0,1)$.

(b) Let the coordinates of D be (x,y).

$$\begin{aligned} \text{Midpoint of } AC &= \text{Midpoint of } BD \qquad (\because ABCD \text{ is a prallelogram.}) \\ (0,1) &= \left(\frac{2+x}{2}, \frac{4+y}{2}\right) \end{aligned}$$

Equating the x-coordinates and y-coordinates respectively, we have

$$\frac{2+x}{2} = 0 \quad \text{and} \quad \frac{4+y}{2} = 1$$

$$2+x = 0 \quad \text{and} \quad 4+y = 2$$

$$x = -2 \quad \text{and} \quad y = -2.$$

$$\therefore D(x,y) = (-2,-2).$$

Exercise 1.1

1. Draw a set of coordinate axes. Locate the points, $A(2,3)$, $B(2,-4)$ and $C(-4,3)$. Lable each point with its coordinates. Determine whether each of the line segments AB, BC and CA is horizontal or vertical.

2. Find the missing coordinates in the following table if M is the midpoint of points P and Q.

P	Q	M
$(2,6)$		$(3,3)$
$(3,2)$	$(-3,-1)$	
	$(0,-1)$	$(-3,2)$
$(1,5)$		$(2.5,3.5)$

3. Find the coordinates of the midpoint and the length of the line segment joining these pairs of points.
(a) $(0,0)$ and $(4,-4)$ (b) $(1,5)$ and $(3,1)$ (c) $(-3,-3)$ and $(0,0)$
(d) $(-1,3)$ and $(5,1)$ (e) $(-1,6)$ and $(2,-2)$ (f) $(-3,-4)$ and $(3,-1)$

4. If $(1, 0)$ is the midpoint of the line passing through the points $A(-5, 2)$ and $B(x, y)$, find the value of x and of y.

5. Calculate the perimeter of given polygons correct to one decimal place.
   (a) A triangle with vertices $P(-2, 3), Q(5, -4)$ and $R(1, 8)$.
   (b) A parallelogram with vertices $A(-10, 1), B(6, -2), C(14, 4)$ and $D(-2, 7)$.
   (c) A trapezium with vertices $E(-6, -2), F(1, -2), G(0, 4)$ and $H(-5, 4)$.

6. A circle has centre (2, 1). Find the coordinates of the endponit of a diameter if one endpoint is $(7, 1)$.

7. $\triangle KLM$ has vertices $K(-5, 18), L(10, -2)$ and $M(-5, -10)$.
   (a) Find the length of each side.
   (b) Find the perimeter of $\triangle KLM$.
   (c) Find the area of $\triangle KLM$.

8. Prove that the triangle whose vertices are $P(2, 3)$, $Q(-1, -1)$, $R(3, -4)$ is isosceles.

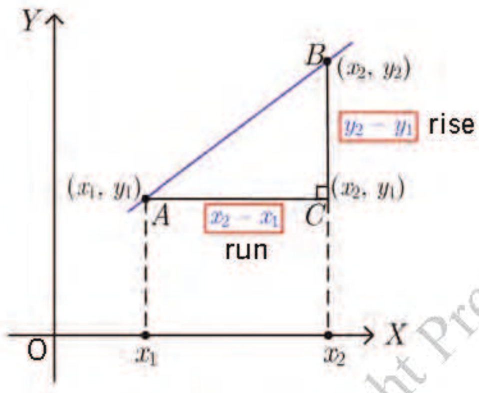
9. A triangle has vertices $E(0, 7), F(5, -5)$ and $G(10, 7)$. Find the length of the altitude to the shortest side.

10. The vertices of a quadrilateral are $A(4, -3), B(7, 10), C(-8, 2)$ and $D(-1, -5)$. Find the length of each diagonal.

11. The distance between the two points $M(15, a)$ and $N(a, -5)$ is 20. Find the value of a.

12. In a parallelogram $PQRS$, three of vertices are $P(1, 1)$, $Q(2, 6)$ and $R(5, 3)$. Find the midpoint of PR and use it to find the fourth vertex S. Find also the lengths of the diagonals.

1.2 Slope of a Straight Line



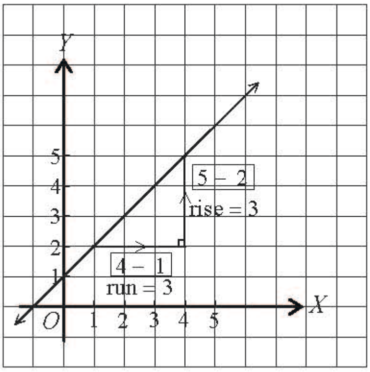
The slope of the line passing through any two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is a measure of the steepness of the line AB. Simply it is the ratio of the vertical change (rise) divided by the horizontal change (run). To determine rise and run, select any two points on the line. The horizontal distance between these two points is called the run and the vertical distance is called the rise.

$$\text{slope m} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{rise}}{\text{run}} = \frac{BC}{AC} = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{or} \quad \frac{y_1 - y_2}{x_1 - x_2}.$$

> **Slope Formula:**
> The slope of the line passing through any two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is
> $$\text{Slope m} = \frac{y_2 - y_1}{x_2 - x_1}.$$

For example, to find the slope of a slanting line, select any two points on a line to determine the rise and run.

(i) Positive Slope



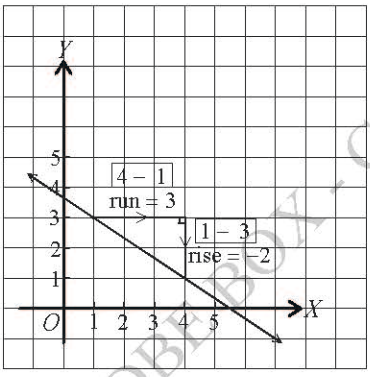
create a right-angled triangle to determine rise and run

increase in x, run $= 3$

increase in y, rise $= 3$

slope $= \dfrac{\text{rise}}{\text{run}} = \dfrac{3}{3} = 1$

(ii) Negative Slope

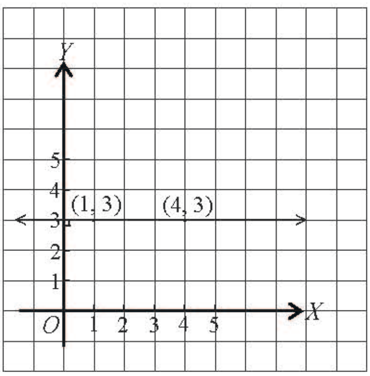


increase in x, run $= 3$

decrease in y, rise $= -2$

slope $= \dfrac{\text{rise}}{\text{run}} = \dfrac{-2}{3} = -\dfrac{2}{3}$

(iii) Zero Slope



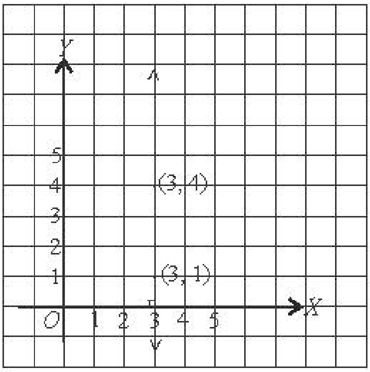
increase in x, run $= 3$

decrease in y, rise $= 0$

slope $= \frac{\text{rise}}{\text{run}} = \frac{3-3}{4-1} = \frac{0}{3} = 0$

Note that the slope of the horizontal line is zero.

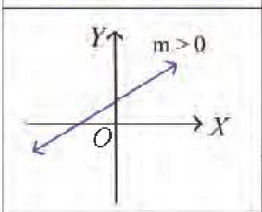
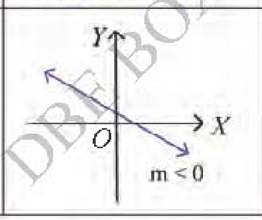
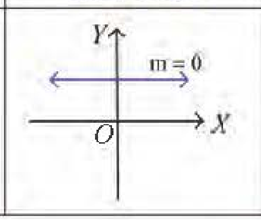
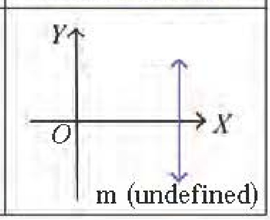
(iv) Undefined Slope



increase in x, run $= 0$
decrease in y, rise $= 3$
slope $= \frac{\text{rise}}{\text{run}} = \frac{4-1}{3-3} = \frac{3}{0}$ (undefined)
Since the denominator is zero,
so the ratio is undefined.

Note that the vertical line has undefined slope.

As shown in the examples, slope can be positive, negative, zero or undefined. By looking at the graph of a line, you may know these cases without calculation. The following table will help you.

Positive Slope	Negative Slope	Zero Slope	Undefined Slope
			

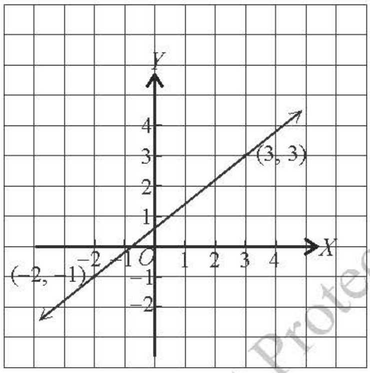
Example 7.
Find the slope of the line AB that passes through the points $A(4, -2)$ and $B(-1, 2)$.

Solution
Let (x_1, y_1) be $(4, -2)$ and (x_2, y_2) be $(-1, 2)$.

$$\text{Slope } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-(-2)}{-1-4} = -\frac{4}{5}.$$

The slope of the line AB is $-\frac{4}{5}$.

Example 8.
Find the slope of the given line.



Solution

Let (3, 3) be (x_1, y_1) and (−2, −1) be (x_2, y_2).

$$\text{Slope } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-3}{-2-3} = \frac{-4}{-5} = \frac{4}{5}.$$

Example 9.
Find the slope of the line that passes through the following points.

x	0	1	2	3
y	5	5	5	5

Solution

Let us choose any two points on the line from the table.
Let (0, 5) be (x_1, y_1) and (2, 5) be (x_2, y_2).

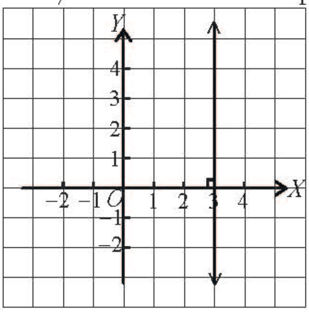
$$\text{Slope } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-5}{2-0} = \frac{0}{2} = 0.$$

Exercise 1.2

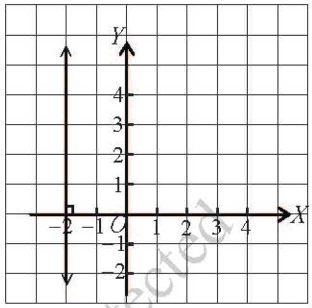
1. Complete each sentence.
   (a) The slope of the line passing through two points $(-6, 0)$ and $(2, 3)$ is ———.
   (b) The slope of the line joining the point (1, 2) and the origin is ———.
   (c) A vertical line has ——— slope.
   (d) A horizontal line has ——— slope.

2. For each graph state whether the slope is positive, negative, zero or undefined, then find the slope if possible.

(a)



(b)



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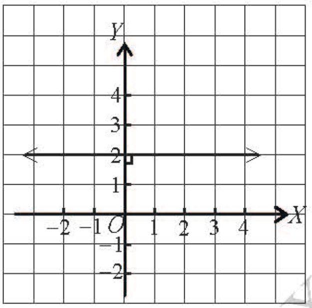

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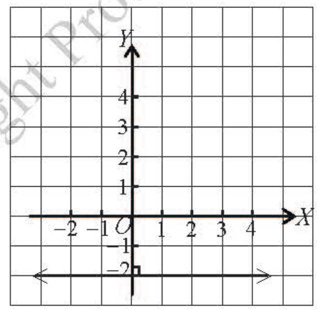

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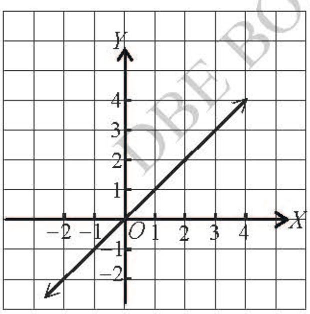

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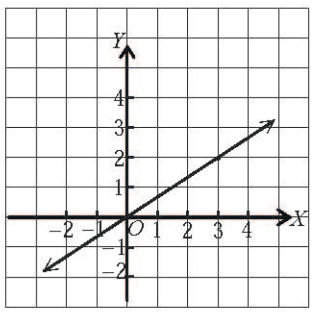

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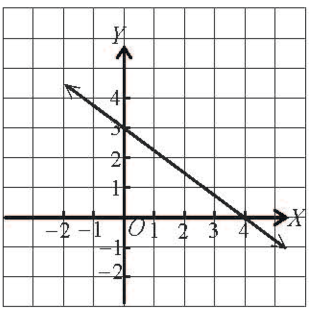

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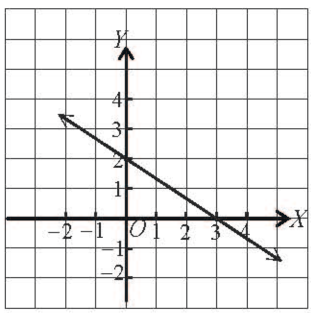


3. Which pairs of points given below will determine horizontal lines? Which ones vertical lines? Determine the slope of each line without calculation.
(a) (5, 2) and $(-3, 2)$ (b) (0, 5) and $(-1, 5)$ (c) (2, 3) and (2, 6)
(d) (0, 0) and $(0, -2)$ (e) $(1, -2)$ and $(-3, -2)$ (f) (a, b) and (a, c)

4. Find the slope of each line which contains each pair of points listed below.
(a) $A(0, 0)$ and $B(8, 4)$ (b) $C(10, 5)$ and $D(6, 8)$
(c) $E(-5, 7)$ and $F(-2, -4)$ (d) $G(23, 15)$ and $H(18, 5)$
(e) $I(-2, 0)$ and $J(0, 6)$ (f) $K(15, 6)$ and $L(-2, 23)$

5. Find the slope of each line which contains each pair of points listed below.
(a) $E(\frac{3}{4}, \frac{4}{5})$ and $F(-\frac{1}{2}, \frac{7}{5})$ (b) $G(-a, b)$ and $H(3a, 2b)$
(c) $L(\sqrt{12}, \sqrt{18})$ and $M(\sqrt{27}, \sqrt{8})$ (d) $P(0, a)$ and $Q(a, 0)$

6. Find p, q, r in the followings:
(a) The slope joining the points $(0, 3)$ and $(1, p)$ is 5.
(b) The slope joining the points $(-2, q)$ and $(0, 1)$ is -1.
(c) The slope joining the points $(-4, -2)$ and $(r, -6)$ is -6.

7. Find the slope corresponding to the following events.
(a) A man climbs 10 m for every 200 meters horizontally.
(b) A motorbike rises 3 m for every 10 meters horizontally.
(c) A plane takes off 1 km for every 5 kilometers horizontally.
(d) A submarine descends 120 m for every 15 meters horizontally.

8. A train climbs a hill with slope 0.05. How far horizontally has the train travelled after rising 15 meters?

9. The vertices of a triangle are the points $A(-2, 3), B(5, -4)$ and $C(1, 8)$. Find the slope of each side.

10. The vertices of a parallelogram are the points $P(1, 4), Q(3, 2), R(4, 6)$ and $S(2, 8)$. Find the slope of each side.

11. A line having a slope of -1 contains the point $(-2, 5)$. What is the y-coordinate of the point on that line whose x-coordinate is 8?

1.3 Lines in the Coordinate Plane

Graph of a Linear Equation

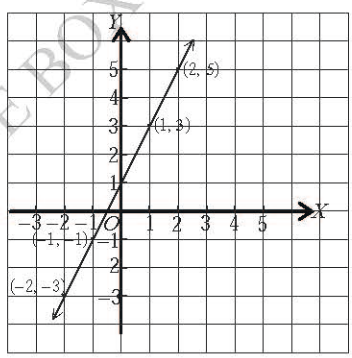
A linear equation is an algebraic equation, in which each term has an exponent of one. To draw the graph of a linear equation $y = mx + c$, you need to plot the coordinates of the points on the line. Construct a table of values of x and y. Choose any two or more values of x within the given interval. If the interval is not given you may choose any values of x. To find the value of y, substitute each value of x in the given linear equation. x and y are the variables in the equation, which means you may take any values. Then plot the points. You will get a straight line graph.

For example, we consider the graph of $y = 2x + 1$.

To draw the given graph, you may choose any the real values of x. Work out the y values and put them in a table.

x	-2	-1	0	1	2
y	-3	-1	1	3	5

Plot the points in the coordinate plane.



To find the slope of given linear equation $y = 2x + 1$, take any points on the line.

If you choose (x_1, y_1) be $(1, 3)$ and (x_2, y_2) be $(2, 5)$, then

$$\text{slope } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{2-1} = \frac{2}{1} = 2.$$

If you choose (x_1, y_1) be $(-2, -3)$ and (x_2, y_2) be $(1, 3)$, then

$$\begin{aligned}\text{slope } m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{3-(-3)}{1-(-2)} = \frac{6}{3} = 2.\end{aligned}$$

Now you can see how the slope of this straight line is related to the equation $y = 2x + 1$. Notice that the slope of the line is equal to the coefficient of x in the given linear equation $y = mx + c$. The point where the line cuts the Y-axis (y-intercept) has a y coordinate that is equal to the constant term in the equation.

Slope-Intercept Form

The equation of the form $y = mx + c$ is the equation of a straight line with slope m and y-intercept c, which is called the slope-intercept form.

Point-Slope Form

We can also find the equation of a line if we are given a point on the line and its slope.

Consider a line with slope m, and passes through the known point $A(x_1, y_1)$. Let $P(x, y)$ be any point on the line. Then

$$m = \frac{y - y_1}{x - x_1}.$$

$$y - y_1 = m(x - x_1).$$

This is the equation of a straight line, with slope m, and passes through the point (x_1, y_1) which is called the point-slope form.

Example 10.

Find the y-intercept and the slope of each line.

(a) $y - 3x + 4 = 0$ (b) $y + 5x = 1$ (c) $x + y = 8$

Solution

equation	$y = mx + c$	slope	y-intercept
$y - 3x + 4 = 0$	$y = 3x - 4$	3	-4
$y + 5x = 1$	$y = -5x + 1$	-5	1
$x + y = 8$	$y = -x + 8$	-1	8

Example 11.

Find the equation of the line with slope -5 and passes through the point $(2,0)$ and draw the graph.

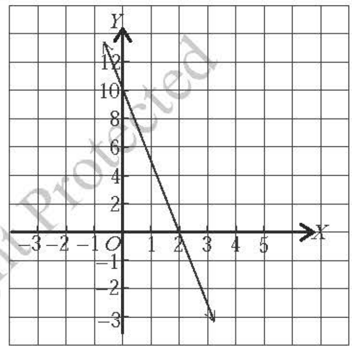
Solution

The equation of the line is $y = -5x + c$, where c is a constant.

Since the point (2, 0) is on the line, substituting $x = 2$ and $y = 0$ in the equation,

$$0 = -5(2) + c$$
$$c = 10.$$

Thus the equation of the line is $y = -5x + 10$.



Alternative Method:

Since the point (2, 0) is on the line, substituting $x_1 = 2$ and $y_1 = 0$, and slope $m = -5$ in point-slope form,

$$y - y_1 = m(x - x_1)$$
$$y - 0 = -5(x - 2)$$
$$y = -5x + 10.$$

Thus the equation of the line is $y = -5x + 10$.

Example 12.

Find the equation of the line passing through the points $A(1,3)$ and $B(4,9)$. Find also the y-intercept.

Solution

Let (x_1, y_1) be $(1,3)$ and (x_2, y_2) be $(4,9)$.

$$\text{slope } m = \frac{9-3}{4-1} = \frac{6}{3} = 2.$$

$\therefore$ the equation of the line is $y = 2x + c$, where c is y-intercept.

Since the line passing through (1, 3), substituting $x = 1$ and $y = 3$ in the equation,

$$y = 2x + c$$
$$3 = 2(1) + c$$
$$c = 1.$$

Thus the equation of the line is $y = 2x + 1$ and the y-intercept is 1.

Alternative Method:
Let (x_1, y_1) be $(1, 3)$ and (x_2, y_2) be $(4, 9)$.

$$\text{slope } m = \frac{9-3}{4-1} = \frac{6}{3} = 2.$$

Since the point (1, 3) is on the line, substituting $x = 1$ and $y = 3$, and slope $m = 2$ in point-slope form,

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 3 &= 2(x - 1) \\ y &= 2x + 1. \end{aligned}$$

Thus the equation of the line is $y = 2x + 1$, and the y-intercept is 1.

Example 13.
The coordinates of the points A and B are $(2, 5)$ and $(-1, 5)$ respectively. Find the equation of the line AB.
Solution
Let (x_1, y_1) be $(2, 5)$ and (x_2, y_2) be $(-1, 5)$.

$$\text{slope } m = \frac{5-5}{-1-2} = \frac{0}{-3} = 0.$$

Equation of AB is of the form $y = 0x + c$. Since the point (2, 5) is on the line, substituting $x = 2$ and $y = 5$ into the equation,
$5 = 0 \times 2 + c$
$c = 5$.
Thus the equation of the line is $y = 5$.

Alternative Method:
The point (2, 5) is on the line. Substituting $x = 2$ and $y = 5$, and slope $m = 0$ in point-slope form,

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 5 &= 0(x - 2) \\ y &= 5. \end{aligned}$$

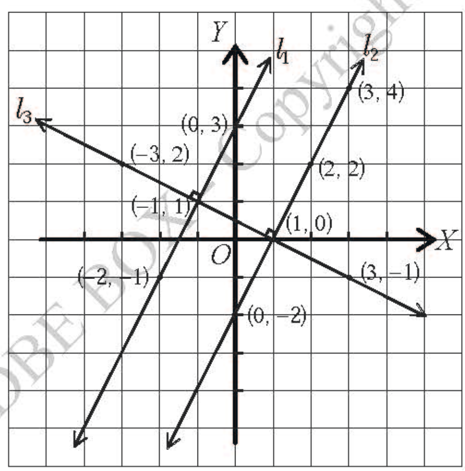
Thus the equation of the line is $y = 5$.

Note that the equation of the horizontal line is $y = a$, $a =$ constant and the equation of the vertical line is $x = b$, $b =$ constant. On a line, all segments have the same slope. Points on the same straight line are said to be collinear.

Parallel and Perpendicular Lines

Any two horizontal lines are parallel. Any two vertical lines are parallel. Vertical line and horizontal line are perpendicular. *Two non-vertical lines are parallel if and only if they have the same slope. Two non-vertical lines are perpendicular if and only if the product of their slopes is -1 (i.e., one is the negative reciprocal of the other).*

For example, in the following figure, the lines l_1 and l_2 are parallel, and the lines l_1 and l_2 are perpendicular to the line l_3.



$$\text{slope of line } l_1 = m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-1}{0-(-1)} = \frac{2}{1} = 2$$

$$\text{slope of line } l_2 = m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-2}{3-2} = \frac{2}{1} = 2$$

$$\text{slope of line } l_3 = m_3 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-2}{3-(-3)} = \frac{-3}{6} = -\frac{1}{2}$$

$\therefore m_1 = m_2,\ m_1 m_3 = m_2 m_3 = -1.$

Exercise 1.3

1. Sketch the following lines.
   (a) $y = 3$ (b) $x = -2$ (c) $y = 5x$ (d) $y = -3x$ (e) $y = \frac{1}{2}x$

2. Graph each line in the same coordinate plane.
   (a) $y = x + 2$ (b) $y = x + 3$ (c) $y = x + 5$
   (d) $y = x - 1$ (e) $y = x - 2$ (f) $y = x - 4$

3. Graph each line.
   (a) $y = 2x + 2$ (b) $y = \frac{1}{2}x + 2$ (c) $y = -\frac{1}{2}x + 2$

4. Find the slope and y-intercept for the following equations and sketch their graphs.
   (a) $y = x - 2$ (b) $y = -3x - 3$ (c) $y = \frac{1}{2}x + 1$
   (d) $y = -\frac{1}{2}x + 1$ (e) $y + 3x = 3$ (f) $x - y = 3$

5. Find the equation of the straight line with the given slope and y-intercept.
   (a) slope 3, y-intercept 4 (b) slope 2, y-intercept 0
   (c) slope 0, y-intercept 2 (d) slope 0, y-intercept 0

6. Find the equation of the line which has a slope m of $-\frac{2}{3}$ and passes through the point (9, 4).

7. A line has slope -2 and y-intercept 6, find its x-intercept.

8. Find the equation of the line which:
   (a) has a slope of 5 and passes through the point (2, 9)
   (b) has a slope of 1 and passes through the point (1, -2)
   (c) has a slope of -3 and passes through the point $(-1, 6)$
   (d) has a slope of -2 and passes through the point $(-1, 4)$.

9. Find the slope and equation of the line joining the following pairs of points.
   (a) (2, 4) and (6, 8) (b) $(-3, 5)$ and $(6, -1)$ (c) $(-2, 1)$ and $(-4, -2)$

10. Determine which of the pairs of lines in each case with given equations are parallel or perpendicular or neither.
 (a) $y = 3x - 2$ and $y = 3x + 9$ (b) $y = \frac{2}{3}x - 5$ and $y = \frac{3}{2}x - 5$
 (c) $y = 3x - 2$ and $y = -\frac{1}{3}x + 9$ (d) $y = \frac{2}{3}x - 5$ and $y = -\frac{3}{2}x - 5$
11. Find the equation of the line which is parallel to the line:
 (a) with equation $y = 4x + 2$ and passes through (0, 8)
 (b) with equation $y = -x + 3$ and passes through (0, 5)
 (c) with equation $y = -2x - 3$ and passes through $(0, -7)$
 (d) with equation $y = -\frac{4}{5}x - 3$ and passes through $(0, \frac{1}{2})$
12. Find the equation of the line which is perpendicular to the line:
 (a) with equation $y = 5x - 4$ and passes through (0, 7)
 (b) with equation $y = -x + 7$ and passes through (0, 4)
 (c) with equation $y = -2x + 3$ and passes through $(0, -4)$
 (d) with equation $y = x - \frac{3}{2}$ and passes through $(0, \frac{5}{4})$
13. Show that the line through (3n, 0) and (0, 7n) is parallel to the line through (0, 21n) and (9n, 0).
14. Prove that the triangle whose vertices are $H(-12,1), K(9,3)$ and $M(11,-18)$ is a right triangle.
15. Given the points $P(1,2), Q(5,-6)$ and $R(b,b)$, determine the value of b so that angle PQR is a right angle.
16. A right-angled isosceles triangle has vertices at (0, 5), (5, 0) and $(-5, 0)$. Find the equation of each of the three sides.
17. Determine the slope of each side of the quadrilateral whose vertices are $A(5,6), B(13,6), C(11,2)$ and $D(1,2)$. Can you tell what kind of a quadrilateral it is?
18. Given the points $D(-4,6), E(1,1)$, and $F(4,6)$, find the slopes of DE and EF. Are the points D, E and F collinear, explain why?
19. Prove that the quadrilateral with vertices $A(-2,2), B(2,-2), C(4,2)$ and $D(2,4)$ is a trapezoid with perpendicular diagonals.
20. Find the slopes of the six lines determined by the points $A(-5,4)$, $B(3,5), C(7,-2), D(-1,-3)$. Prove that $ABCD$ is a rhombus.

Chapter 2

Exponents and Radicals

Exponents are mathematical shorthand that tells us to multiply the same number by itself for a specific number of times. In this chapter, you will learn about positive integral exponents, zero exponent, negative integral exponents, rational exponents, radicals and exponential equations.

2.1 Exponents

Simply stated exponents are shorthand for repeated multiplication of the same element by itself. The exponent corresponds to the number of times the base is used as a factor. For instance, the shorthand for multiplying two copies of three is such as 3^2. Such products of repeated factors are called powers. Powers can be expressed as expanded form or factor form.

In the example $3 \times 3 = 3^2$, the exponent is 2 and base is 3. 3^2 can be also read as **3 to the second power** or **3 squared**.

2.1.1 Positive Integral Exponents

Integral here means "**integer**". So the exponent (or power) is an integer. In a given product, factors may occur more than once.

In writing a product, a raised dot $(\cdot)$ is often used instead of cross $(\times)$ to denote multiplication.

Thus,

$3 \times 3 = 3 \cdot 3 = 3^2$
$5 \times 5 \times 5 = 5 \cdot 5 \cdot 5 = 5^3$
$a \times a \times a \times a = a \cdot a \cdot a \cdot a = a^4$, and so on.

In general, if a is any real number and n is a positive integer, then the n^{th} power of a is

$$\underbrace{a \times a \times a \times a \times \cdots \times a}_{n\ factors} = a^n$$

where the number a is called the base and n is called the exponent or index.

2.1.2 Zero and Negative Integral Exponents

Definition. For any real number x, if $x \neq 0$, then $x^0 = 1$.

For example,

$$3^0 = 1$$
$$(-4)^0 = 1$$

Note that 0^0 is indeterminate.

Definition. For any real number x, if $x \neq 0$, then $x^{-n} = \frac{1}{x^n}$.

For example,

$$5^{-2} = \frac{1}{5^2} = \frac{1}{25}$$
$$\left(\frac{1}{3}\right)^{-4} = \frac{1}{\left(\frac{1}{3}\right)^4} = \frac{1}{\frac{1}{3^4}} = 3^4$$

Consequently $\frac{1}{x^{-n}} = x^n$.

Example 1.

Evaluate the following:

(i) $\frac{3^{-4}}{4^{-3}}$ (ii) $5^0 - 5x^0 - (5x)^{-1} - (5x)^0$.

Solution

(i) $\frac{3^{-4}}{4^{-3}} = \frac{4^3}{3^4} = \frac{64}{81}$

(ii) $5^0 - 5x^0 - (5x)^{-1} - (5x)^0 = 1 - 5(1) - \frac{1}{5x} - 1 = 1 - 5 - \frac{1}{5x} - 1 = -5 - \frac{1}{5x}$

2.1.3 Rules for Integral Exponents

We assume that x and y be any real numbers and m and n are positive integers.

Rule 1 (Multiplication)

$$\boxed{x^m \cdot x^n = x^{m+n}}$$

For example,

$$\begin{aligned} 3^3 \cdot 3^5 &= 3^{3+5} = 3^8 \\ x^2 \cdot x^6 &= x^{2+6} = x^8 \end{aligned}$$

Rule 2 (Division)

$$\boxed{x^m \div x^n = \tfrac{x^m}{x^n} = \begin{cases} x^{m-n}, & \text{if} \quad m > n \\ 1, & \text{if} \quad m = n \\ \dfrac{1}{x^{n-m}}, & \text{if} \quad m < n, x \neq 0 \end{cases}}$$

For example,

$$\begin{aligned} 3^5 \div 3^2 &= \frac{3^5}{3^2} = 3^{5-2} = 3^3 \\ a^3 \div a^8 &= \frac{a^3}{a^8} = \frac{1}{a^{8-3}} = \frac{1}{a^5} \\ a^4 \div a^4 &= \frac{a^4}{a^4} = 1 \end{aligned}$$

Rule 3 (Power of a Power)

$$\boxed{(x^m)^n = x^{mn}}$$

For example,

$$\begin{aligned}(2^3)^4 &= 2^{3\cdot 4} = 2^{12}\\ (x^4)^2 &= x^{4\cdot 2} = x^8\end{aligned}$$

Rule 4 (Power of a Product)

$$\boxed{(xy)^n = x^n \cdot y^n}$$

For example,

$$\begin{aligned}(2\cdot 3)^4 &= 2^4 \cdot 3^4\\ (xy)^5 &= x^5 \cdot y^5\end{aligned}$$

Rule 5 (Power of a Quotient)

$$\boxed{\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}, \quad y \neq 0}$$

For example,

$$\begin{aligned}\left(\frac{3}{5}\right)^2 &= \frac{3^2}{5^2}\\ \left(\frac{x}{y}\right)^3 &= \frac{x^3}{y^3}, \quad y \neq 0\end{aligned}$$

The following examples are computed by the use of Rule 1 through Rule 5.

Example 2.

Simplify and name the rules used.

(i) $x^7 \cdot x^4$ (ii) $a^{15} \div a^7$ (iii) $(b^2)^3$ (iv) $a^5 \cdot b^5$ (v) $\frac{y^7}{4^7}$

Solution

(i) $x^7 \cdot x^4 = x^{7+4} = x^{11}$ (Multiplication Rule)

(ii) $a^{15} \div a^7 = a^{15-7} = a^8$ (Division Rule)

(iii) $(b^2)^3 = b^6$ (Power of a Power Rule)

(iv) $a^5 \cdot b^5 = (ab)^5$ (Power of a Product Rule)

(v) $\frac{y^7}{4^7} = (\frac{y}{4})^7$ (Power of a Quotient Rule)

Example 3.

Simplify and name the rules used.

(i) $\left(\frac{-81x^3y^4}{27xy^3}\right)^3$ (ii) $(a^{-1}b^{-3})^{-2}$

Solution

(i)
$$\begin{aligned}\left(\frac{-81x^3y^4}{27xy^3}\right)^3 &= (-3x^2y)^3 && \text{(Division Rule)}\\ &= (-3)^3(x^2)^3y^3 && \text{(Power of a Product Rule)}\\ &= -27x^6y^3 && \text{(Power of a Power Rule)}\end{aligned}$$

(ii)
$$\begin{aligned}(a^{-1}b^{-3})^{-2} &= (a^{-1})^{-2}(b^{-3})^{-2} && \text{(Power of a Product Rule)}\\ &= a^2b^6 && \text{(Power of a Power Rule)}\end{aligned}$$

Example 4.

Evaluate $\left(\frac{16 \cdot 27}{25}\right)^2\left(\frac{50}{36}\right)^3$.

Solution

$$\begin{aligned}\left(\frac{16 \cdot 27}{25}\right)^2\left(\frac{50}{36}\right)^3 &= \left(\frac{2^4 \cdot 3^3}{5^2}\right)^2\left(\frac{2 \cdot 5^2}{2^2 \cdot 3^2}\right)^3\\ &= \frac{(2^4 \cdot 3^3)^2}{(5^2)^2}\frac{(2 \cdot 5^2)^3}{(2^2 \cdot 3^2)^3}\\ &= \frac{2^8 \cdot 3^6}{5^4} \cdot \frac{2^3 \cdot 5^6}{2^6 \cdot 3^6}\\ &= 2^5 \cdot 5^2\\ &= 32 \cdot 25\\ &= 800\end{aligned}$$

Example 5.

Evaluate $\dfrac{(x^2-y^2)^3}{(x+y)^3}$.

Solution

$$\begin{aligned}\frac{(x^2-y^2)^3}{(x+y)^3} &= \frac{((x+y)(x-y))^3}{(x+y)^3}\\ &= \frac{(x+y)^3(x-y)^3}{(x+y)^3}\\ &= (x-y)^3\end{aligned}$$

Exercise 2.1

1. Simplify by using the rules of exponents and name the rules used.

   (a) $\dfrac{36a^4b^5}{100a^7b^3}$ (b) $\dfrac{27a^2b^5}{(9a^2b)^2}$ (c) $\left(\dfrac{-135a^4b^5c^6}{315a^6b^7c^8}\right)^2$

   (d) $\left(\dfrac{x^4}{y^5}\right)^3\left(\dfrac{y^3}{x^2}\right)^2$ (e) $\dfrac{2^{3^2}}{(2^2)^3}$

2. Evaluate the following.

   (a) $\dfrac{54^2\times 12^3\times 64^2(3^2\times 4^3\times 5^2)^3}{(3^2\times 15\times 20^3)^4}$ (b) $\left(\dfrac{343}{36}\right)^3\left(\dfrac{540}{56}\right)^4$

   (c) $\left(\dfrac{33}{1056}\right)^3\left(\dfrac{768}{270}\right)^4\left(\dfrac{450}{48}\right)^3$

3. Simplify.

   (a) $\left(\dfrac{3^m}{15^n}\right)^3\left(\dfrac{45^n}{255^m}\right)^2$ (b) $\left(\dfrac{20^x}{400^y}\right)^2\left(\dfrac{150^{y^2}}{180^x}\right)^3$

   (c) $\dfrac{(x^3-y^3)(x+y)}{(x^2-y^2)^3}$ (d) $\dfrac{(x^{a-b}x^{b-c})^a\left(\dfrac{x^a}{x^c}\right)^c}{(x^bx^c)^a\div(x^{a+c})^c}$

4. Evaluate the following.

(a) $(-3)^{-2}$ (b) -3^{-3} (c) $-2^0 + 5^{-1}$

(d) $(-2)^{-3} + 2^{-2} - 2^{-4}$ (e) $5^0 - (-3)^0$ (f) $\frac{27^{-6}}{125^{-3}} \div \frac{9^{-2}}{25^{-4}}$

(g) $(-5)^0 - (-5)^{-1} - (-5)^{-2} - (-5)^{-3}$ (h) $(-1)^{(-1)^{-1}}$

(i) $\frac{(180^2)^{-3}(6 \cdot 90^{-2})^3}{(40^{-3})^2 \cdot 25^{-2}}$ (j) $\frac{(2^{-3} - 3^{-2})^{-1}}{(2^{-3} + 3^{-2})^{-1}}$

5. Simplify the following.

(a) $(-3a^4)(4a^{-7})$ (b) $\left(\frac{2x^{-4}}{5y^2z^3}\right)^{-2}$

(c) $\left(\frac{x^{2m+n}x^{3(m-n)}}{x^{m-2n}x^{2m-n}}\right)^{-3}$ (d) $\left(\frac{2x^{-3}y^2}{3^{-1}y^3}\right)^2\left(\frac{4x^{-2}y^3}{3x^5}\right)^3 \div \left(\frac{81x^{-2}}{y^{-3}}\right)^{-2}$

(e) $\frac{2x+y}{x^{-1}+2y^{-1}}$ (f) $(x^{-2} - y^{-1})^{-3}$

(g) $\frac{x^{-2} - y^{-2}}{x^{-1}+y^{-1}}$ (h) $\frac{(x+y^{-1})^2}{1+x^{-1}y^{-1}}$

2.1.4 Rational Exponents

In this section, you will examine the root of a number and the rational exponents.

Definition. If n is a positive integer and x and y are real numbers, such that $x^n = y$, then x is called the n^{th} root of y.

For example,

(i) $2^3 = 8$

2 is the cube root of 8.

(ii) $(-2)^3 = -8$
-2 is the cube root of -8.
(iii) $(-3)^4 = 81$
-3 is the fourth root of 81.
(iv) $3^4 = 81$
3 is the fourth root of 81.

In general, for $x^n = y$, if n is odd, there is only one real n^{th} root of y, no matter whether y is negative or zero or positive. In this case the real n^{th} root of y is denoted by $\sqrt[n]{y}$ and is called **the principal n^{th} root of y**.

For example,
$\sqrt[3]{8} = 2$ by (i)
$\sqrt[3]{-8} = -2$ by (ii)

If n is even and y is positive, there are two real n^{th} roots of y, one positive and the other negative. In that case the positive n^{th} root of y is denoted by $\sqrt[n]{y}$ and is called **the principal n^{th} root of y**.
For example,
By (iii) and (iv), 3 and -3 are fourth roots of 81 and the principal fourth root of 81, $\sqrt[4]{81} = 3$.

Notice the following:

(i) $x^2 = -4$ has no real number x because the square of any nonzero real number is positive.
(ii) It is important to understand the difference between $\sqrt[n]{-b}$ and $-\sqrt[n]{b}$. $\sqrt[n]{-b}$ is the principal n^{th} root of of $-b$ and $-\sqrt[n]{b}$ is the negative of the principal n^{th} root of b.
(iii) $\sqrt{x^2} = \begin{cases} x & \text{if } x \text{ is positive or zero,} \\ -x & \text{if } x \text{ is negative.} \end{cases}$

For example,
(i) $\sqrt{9} = \sqrt{3^2} = 3$
(ii)$\sqrt{9} = \sqrt{(-3)^2} = -(-3) = 3$

Now we will extend the exponential concept to fractional exponents. First, we consider the exponents of the form $\frac{1}{n}$, where n is a positive integer. Here $b^{\frac{1}{n}}$ is the n^{th} root of b (provided such a root exists). Then we define as follows:

Definition. For a real number x and an integer n $(n \geq 2)$, $x^{\frac{1}{n}} = \sqrt[n]{x}$, when n is even, x must be positive or zero.

Example 6.

Simplify (a)$16^{\frac{1}{2}}$ (b) $8^{\frac{1}{3}}$ (c) $(-27)^{\frac{1}{3}}$.

Solution

(a) $16^{\frac{1}{2}} = \sqrt{16} = 4$

(b) $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$

(c) $(-27)^{\frac{1}{3}} = \sqrt[3]{-27} = -3$

Next, we now consider the more general fractional exponent expression $x^{\frac{m}{n}}$. If m and n are positive integers, to preserve the rules for exponents we want to write

$$(x^{\frac{1}{n}})^m = x^{\frac{m}{n}}.$$

This observation leads to the following definition.

Definition. If m and n are positive integers and $\frac{m}{n}$ is a rational number in lowest terms, then for any real number x,

$$x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m,$$

when n is even, x must be positive or zero.

The following calculation for $(-8)^{\frac{2}{6}}$ is false because the exponent $\frac{2}{6}$ is not in lowest terms.

$$(-8)^{\frac{2}{6}} = \sqrt[6]{(-8)^2} = \sqrt[6]{64} = 2 \quad \times$$

Calculate $(-8)^{\frac{2}{6}}$ as follows:

$$(-8)^{\frac{2}{6}} = (-8)^{\frac{1}{3}} = \sqrt[3]{-8} = -2 \quad \checkmark$$

Definition. If m and n are any positive integers, then for any real number $x \neq 0$, $x^{-\frac{m}{n}} = \frac{1}{x^{\frac{m}{n}}}$.

For example,

(i) $8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{(\sqrt[3]{8})^2} = \frac{1}{2^2} = \frac{1}{4}$

(ii) $32^{-\frac{7}{5}} = \frac{1}{32^{\frac{7}{5}}} = \frac{1}{(\sqrt[5]{32})^7} = \frac{1}{2^7} = \frac{1}{128}$.

Example 7.

Simplify and express the answer with positive exponents $\left(\frac{2^m\sqrt{2^{-m}}}{\sqrt{2^{m+1}}}\right)^{-2m}$.

Solution

$$
\begin{aligned}
\left(\frac{2^m\sqrt{2^{-m}}}{\sqrt{2^{m+1}}}\right)^{-2m} &= \left(\frac{2^m 2^{-\frac{m}{2}}}{2^{\frac{m+1}{2}}}\right)^{-2m} \\
&= \left(2^{m-\frac{m}{2}-\frac{m+1}{2}}\right)^{-2m} \\
&= \left(2^{-\frac{1}{2}}\right)^{-2m} \\
&= 2^m
\end{aligned}
$$

Note that the rules for integral exponents hold for all rational exponents. The following are some useful expressions:

$$
\begin{aligned}
x^2 - y^2 &= (x+y)(x-y) \\
x - y &= (x^{\frac{1}{2}})^2 - (y^{\frac{1}{2}})^2 = (x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^{\frac{1}{2}} - y^{\frac{1}{2}}) \\
x^{\frac{1}{2}} - y^{\frac{1}{2}} &= (x^{\frac{1}{4}})^2 - (y^{\frac{1}{4}})^2 = (x^{\frac{1}{4}} + y^{\frac{1}{4}})(x^{\frac{1}{4}} - y^{\frac{1}{4}}) \\
x^3 - y^3 &= (x-y)(x^2 + xy + y^2) \\
x - y &= (x^{\frac{1}{3}})^3 - (y^{\frac{1}{3}})^3 = (x^{\frac{1}{3}} - y^{\frac{1}{3}})(x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}) \\
x^3 + y^3 &= (x+y)(x^2 - xy + y^2) \\
x + y &= (x^{\frac{1}{3}})^3 + (y^{\frac{1}{3}})^3 = (x^{\frac{1}{3}} + y^{\frac{1}{3}})(x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}})
\end{aligned}
$$

Example 8.

Simplify $\dfrac{x-y^{-1}}{(x^{\frac{1}{2}}+y^{-\frac{1}{2}})(x^{\frac{1}{4}}-y^{-\frac{1}{4}})}$.

Solution

$$\begin{aligned}
\frac{x-y^{-1}}{(x^{\frac{1}{2}}+y^{-\frac{1}{2}})(x^{\frac{1}{4}}-y^{-\frac{1}{4}})} &= \frac{(x^{\frac{1}{2}}+y^{-\frac{1}{2}})(x^{\frac{1}{2}}-y^{-\frac{1}{2}})}{(x^{\frac{1}{2}}+y^{-\frac{1}{2}})(x^{\frac{1}{4}}-y^{-\frac{1}{4}})} \\
&= \frac{(x^{\frac{1}{2}}-y^{-\frac{1}{2}})}{(x^{\frac{1}{4}}-y^{-\frac{1}{4}})} \\
&= \frac{(x^{\frac{1}{4}}+y^{-\frac{1}{4}})(x^{\frac{1}{4}}-y^{-\frac{1}{4}})}{(x^{\frac{1}{4}}-y^{-\frac{1}{4}})} \\
&= x^{\frac{1}{4}}+y^{-\frac{1}{4}} \\
&= x^{\frac{1}{4}}+\frac{1}{y^{\frac{1}{4}}} \\
&= \frac{x^{\frac{1}{4}}y^{\frac{1}{4}}+1}{y^{\frac{1}{4}}}
\end{aligned}$$

Exercise 2.2

1. Evaluate the following.

   (a) $(125)^{\frac{2}{3}}$ (b) $(81)^{-\frac{3}{2}}$ (c) $(-27)^{\frac{2}{3}}$

   (d) $\left(\dfrac{16}{81}\right)^{-\frac{3}{4}}$ (e) $\left(\dfrac{-125}{8}\div\dfrac{1}{64}\right)^{\frac{1}{3}}$ (f) $(0.125)^{-\frac{2}{3}}$

   (g) $\left(\dfrac{64}{27}\right)^{-\frac{2}{3}}$ (h) $(-4)^{-1}+(-1)^{-4}$

2. Simplify the following.

   (a) $\sqrt[3]{4^2} \cdot 4^{\frac{2}{3}} \cdot \left(\frac{1}{4}\right)^{-\frac{2}{3}}$ (b) $\sqrt{\frac{512 \times 27^{-3} \times 81 \times 3^8}{3^4}}$

   (c) $\left(\left(\frac{3}{4}\right)^{-4}\right)^{-0.5} \cdot \sqrt{\left(\frac{4}{3}\right)^{-1}} \div 16^{-0.5}$

   (d) $(27)^{\frac{1}{4}} + \frac{24}{(8)^{-\frac{2}{3}}} + \frac{\sqrt[5]{2}}{(4)^{-\frac{2}{5}}}$ (e) $\frac{(243)^{\frac{4}{5}} + (64)^{\frac{2}{3}} - (216)^{\frac{1}{3}}}{(225)^{\frac{1}{2}} - (16)^{\frac{3}{4}}}$

3. Simplify the following.

   (a) $\frac{x - 5\sqrt{x}}{x - 2\sqrt{x} - 15} \div \left(1 + \frac{3}{\sqrt{x}}\right)^{-1}$ (b) $\sqrt[a]{\frac{\sqrt[b]{x}}{\sqrt[c]{x}}} \cdot \sqrt[b]{\frac{\sqrt[c]{x}}{\sqrt[a]{x}}} \cdot \sqrt[c]{\frac{\sqrt[a]{x}}{\sqrt[b]{x}}}$

   (c) $\left[\frac{x^m - y^m}{x^{\frac{m}{2}} - y^{\frac{m}{2}}} - \frac{x^m - y^m}{x^{\frac{m}{2}} + y^{\frac{m}{2}}}\right]^{-2}$

2.2 Radicals

In above section, the principal n^{th} root of b is denoted by $\sqrt[n]{b}$ and also written as $b^{\frac{1}{n}}$. There is another name for roots of a number.

Definition. The symbol $\sqrt[n]{b}$ is called a **radical**, $\sqrt[n]{\ }$ is the **radical sign**, n is the **order or index**, and b is called the **radicand**.

2.2.1 Rules for Radicals

For any positive integers m, n and k,

Rule 1 $\boxed{\left(\sqrt[n]{x}\right)^n = \sqrt[n]{x^n} = x}$

For example,

$$\left(\sqrt[3]{5}\right)^3 = 5$$
$$\sqrt[4]{b^4} = b$$

Rule 2 $\boxed{\sqrt[n]{x} \cdot \sqrt[n]{y} = \sqrt[n]{xy}}$

For example,

$$\begin{aligned} \sqrt[3]{5} \cdot \sqrt[3]{20} &= \sqrt[3]{5 \cdot 20} = \sqrt[3]{100} \\ \sqrt[4]{x^2} \cdot \sqrt[4]{y} &= \sqrt[4]{x^2 y} \end{aligned}$$

Rule 3 $\boxed{\sqrt[m]{\sqrt[n]{x}} = \sqrt[mn]{x}}$

For example,

$$\begin{aligned} \sqrt{\sqrt[3]{5}} &= \sqrt[6]{5} \\ \sqrt[3]{\sqrt[4]{x^5}} &= \sqrt[12]{x^5} \end{aligned}$$

Rule 4 $\boxed{\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}, \qquad y \neq 0}$

For example,

$$\begin{aligned} \sqrt[3]{\frac{5}{9}} &= \frac{\sqrt[3]{5}}{\sqrt[3]{9}} \\ \sqrt[4]{\frac{x^5}{y^3}} &= \frac{\sqrt[4]{x^5}}{\sqrt[4]{y^3}} \end{aligned}$$

Rule 5

$$\boxed{\begin{array}{l} \text{(a) } \sqrt[n]{x^m} = \sqrt[kn]{x^{km}} \\ \text{(b) } \sqrt[n]{x^m} = \sqrt[\frac{n}{k}]{x^{\frac{m}{k}}}, \quad k \neq 0 \end{array}}$$

For example,

$$\begin{aligned} \sqrt[3]{5} &= \sqrt[3\times 2]{5^2} = \sqrt[6]{5^2} \\ \sqrt[5]{b^3} &= \sqrt[5\times 3]{b^{3\times 3}} = \sqrt[15]{b^9} \\ \sqrt[6]{25} &= \sqrt[6]{5^2} = \sqrt[3]{5} \\ \sqrt[12]{b^3} &= \sqrt[4]{b} \end{aligned}$$

Example 9.

Simplify the following.
(a) $\sqrt{27}$ (b) $\sqrt[3]{16}$ (c) $\sqrt{\sqrt[3]{128}}$ (d) $\sqrt[5]{-32}$ (e) $\sqrt{72}$

Solution

(a) $\sqrt{27} = \sqrt{9 \times 3} = \sqrt{3^2 \times 3} = 3\sqrt{3}$

(b) $\sqrt[3]{16} = \sqrt[3]{8 \times 2} = \sqrt[3]{2^3 \times 2} = 2\sqrt[3]{2}$

(c) $\sqrt{\sqrt[3]{128}} = \sqrt[6]{128} = \sqrt[6]{2^6 \times 2} = 2\sqrt[6]{2}$

(d) $\sqrt[5]{-32} = \sqrt[5]{(-2)^5} = -2$

(e) $\sqrt{72} = \sqrt{36 \times 2} = \sqrt{6^2 \times 2} = 6\sqrt{2}$

Example 10.

Change the expression with the same radical and simplify the radicands.

(a) $5a\sqrt[3]{3}$ (b) $\sqrt[3]{5}\sqrt{3}$ (c) $\sqrt[4]{2}\sqrt[3]{3}$ (d) $\sqrt{2x}\sqrt[3]{3y}$

Solution

(a) $5a\sqrt[3]{3} = \sqrt[3]{(5a)^3}\sqrt[3]{3} = \sqrt[3]{125a^3}\sqrt[3]{3} = \sqrt[3]{375a^3}$

(b) $\sqrt[3]{5}\sqrt{3} = \sqrt[6]{5^2}\sqrt[6]{3^3} = \sqrt[6]{25 \times 27} = \sqrt[6]{675}$

(c) $\sqrt[4]{2}\sqrt[3]{3} = \sqrt[12]{2^3}\ \sqrt[12]{3^4} = \sqrt[12]{8 \times 81} = \sqrt[12]{648}$

(d) $\sqrt{2x}\sqrt[3]{3y} = \sqrt[6]{(2x)^3}\sqrt[6]{(3y)^2} = \sqrt[6]{8x^3 \times 9y^2} = \sqrt[6]{72x^3y^2}$

Exercise 2.3

1. Write the following in radical form.

   (a) $(5)^{\frac{1}{2}}$ (b) $(-9)^{\frac{1}{3}}$ (c) $(2)^{-\frac{1}{2}}$ (d) $\left(-\frac{3}{4}\right)^{\frac{2}{5}}$ (e) $\left(\frac{2}{7}\right)^{\frac{5}{2}}$

2. Write the following in fractional exponent form.

   (a) $\sqrt[6]{c^5}$ (b) $\sqrt[3]{-2}$ (c) $\sqrt[5]{a^4\sqrt[3]{b^5}}$ (d) $\sqrt[4]{\left(\frac{3}{7}\right)^3}$

3. Change the expression with the same radical and simplify the radicands.

   (a) $6\sqrt{2}$ (b) $3a\sqrt[3]{x}$ (c) $2\sqrt[5]{2}$ (d) $3\sqrt[4]{\frac{1}{2}}$ (e) $3\sqrt{x^3}$

4. Simplify.

   (a) $\sqrt{32}$ (b) $\sqrt[5]{-32}$ (c) $\sqrt[4]{\frac{81x^{16}}{16y^4}}$

   (d) $\sqrt[3]{\frac{81x^2}{4y}}$ (e) $\frac{9^{\frac{1}{2}}}{\sqrt[3]{27}}$ (f) $\sqrt{\frac{2}{3}}\cdot\sqrt{\frac{75}{98}}$

   (g) $\sqrt[3]{\frac{-216}{8\times 10^3}}$ (h) $\sqrt[n]{\frac{32}{2^{5+n}}}$

5. Rationalize the denominators.

   (a) $\frac{4\sqrt{35}}{3\sqrt{7}}$ (b) $\frac{20}{\sqrt{5}}$ (c) $\frac{18}{\sqrt[3]{2}}$

   (d) $\frac{\sqrt[3]{32}}{\sqrt[4]{27}}$ (e) $\frac{\sqrt[3]{36a^2}}{\sqrt[3]{9a}}$ (f) $\frac{\sqrt[3]{2}}{\sqrt[6]{12}}$

   (g) $\frac{1}{\sqrt[3]{xy^2}}$ (h) $\sqrt[m]{\frac{2x^2y^{3m}}{9x^5y^{4m-1}}}$

6. Reduce the order as far as possible.

   (a) $\sqrt[4]{25}$ (b) $\sqrt[6]{4}$ (c) $\sqrt[6]{8}$

   (d) $\sqrt[9]{8y^3}$ (e) $\sqrt[6]{27^3}$ (f) $\sqrt[8]{a^2b^4}$

   (g) $\sqrt[12]{64a^2b^6}$ (h) $(72)^{\frac{3}{5}}$ (i) $\sqrt[3]{768}$

7. Find the simplified forms.

   (a) $\sqrt{\frac{9}{50}}$ (b) $\sqrt[3]{\frac{-192}{49}}$ (c) $\sqrt[4]{16}$ (d) $2\sqrt[3]{56}$

2.3 Operations with Radicals

The product of two radicals of the same order is found directly from **Rule 2** of radicals. The product of two radicals of different orders can be found using the rule after they have been changed to the equivalent radicals of the same order.

Example 11.
Multiply $\sqrt{6}$ by $\sqrt{15}$ and express in the simplest form.

Solution

$$\sqrt{6}\cdot\sqrt{15} = \sqrt{6\cdot 15} = \sqrt{2\cdot 3\cdot 3\cdot 5} = 3\sqrt{10}$$

Example 12.

Multiply $\sqrt[3]{12}$ by $\sqrt{10}$.

Solution

$$\begin{aligned}\sqrt[3]{12}\cdot\sqrt{10} &= \sqrt[6]{12^2}\cdot\sqrt[6]{10^3}\\ &= \sqrt[6]{12^2\cdot 10^3}\\ &= \sqrt[6]{(3\cdot 2^2)^2\cdot(2\cdot 5)^3}\\ &= \sqrt[6]{3^2\cdot 2^4\cdot 2^3\cdot 5^3}\\ &= \sqrt[6]{3^2\cdot 2^6\cdot 2\cdot 5^3}\\ &= 2\sqrt[6]{2250}\end{aligned}$$

Example 13.
Simplify $\sqrt{3}(\sqrt{3}+\sqrt{8})$.
Solution

$$\begin{aligned}\sqrt{3}(\sqrt{3}+\sqrt{8}) &= \sqrt{3}\cdot\sqrt{3}+\sqrt{3}\cdot\sqrt{8}\\ &= 3+\sqrt{24}\\ &= 3+\sqrt{2^2\cdot 6}\\ &= 3+2\sqrt{6}\end{aligned}$$

The radicals with the same index and same radicand are called similar radicals and they can be added or subtracted. The following example shows the addition and subtraction of the radicals.

Example 14.

Simplify $\sqrt{200}+\sqrt{50}-\sqrt{18}$.
Solution

$$\begin{aligned}\sqrt{200}+\sqrt{50}-\sqrt{18} &= \sqrt{2\cdot 100}+\sqrt{2\cdot 25}-\sqrt{2\cdot 9}\\ &= 10\sqrt{2}+5\sqrt{2}-3\sqrt{2}\\ &= (10+5-3)\sqrt{2}\\ &= 12\sqrt{2}\end{aligned}$$

The algebraic sum of dissimilar radicals can be simplified by rationalizing the denominators.

Example 15.

Simplify $\sqrt{24}+\sqrt{\frac{2}{3}}-\sqrt[3]{\frac{2}{9}}$.
Solution

$$\begin{aligned}\sqrt{24}+\sqrt{\frac{2}{3}}-\sqrt[3]{\frac{2}{9}} &= \sqrt{2^2\cdot 6}+\frac{\sqrt{2}}{\sqrt{3}}-\frac{\sqrt[3]{2}}{\sqrt[3]{3^2}}\\ &= 2\sqrt{6}+\frac{\sqrt{2}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}-\frac{\sqrt[3]{2}}{\sqrt[3]{3^2}}\cdot\frac{\sqrt[3]{3}}{\sqrt[3]{3}}\\ &= 2\sqrt{6}+\frac{\sqrt{6}}{3}-\frac{\sqrt[3]{6}}{\sqrt[3]{3^3}}\\ &= 2\sqrt{6}+\frac{\sqrt{6}}{3}-\frac{\sqrt[3]{6}}{3}\\ &= (2+\frac{1}{3})\sqrt{6}-\frac{\sqrt[3]{6}}{3}\\ &= \frac{7}{3}\sqrt{6}-\frac{1}{3}\sqrt[3]{6}\end{aligned}$$

Example 16.

Simplify $\dfrac{\sqrt{2}-3\sqrt{3}}{2\sqrt{2}-\sqrt{3}}$.

Solution

$$\begin{aligned}\frac{\sqrt{2}-3\sqrt{3}}{2\sqrt{2}-\sqrt{3}} &= \frac{\sqrt{2}-3\sqrt{3}}{2\sqrt{2}-\sqrt{3}}\cdot\frac{2\sqrt{2}+\sqrt{3}}{2\sqrt{2}+\sqrt{3}}\\ &= \frac{4+\sqrt{6}-6\sqrt{6}-9}{4\times 2-3}\\ &= \frac{-5-5\sqrt{6}}{5}\\ &= \frac{-5(1+\sqrt{6})}{5}\\ &= -(1+\sqrt{6})\end{aligned}$$

In rationalizing the denominator, we multiply it by its conjugate. This process is based on the fact that

$$(a+b)(a-b)=a^2-b^2.$$

Each of the two factors is called **the conjugate of the other**. Thus

(1) $a+\sqrt{b}$ and $a-\sqrt{b}$

(2) $a+b\sqrt{c}$ and $a-b\sqrt{c}$

(3) $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$

(4) $a\sqrt{b}+c\sqrt{d}$ and $a\sqrt{b}-c\sqrt{d}$

are conjugate radicals.

Exercise 2.4

1. Simplify the following.

   (a) $3\sqrt{5}+7\sqrt{5}$ (b) $\sqrt{75}-\sqrt{12}$ (c)$3\cdot 3\sqrt{3}\cdot 3\sqrt{27}$

   (d) $2\sqrt{5}\cdot 3\sqrt{2}$ (e)$(4-\sqrt{3})^2$ (f) $(\sqrt{3}+2\sqrt{2})(\sqrt{3}+\sqrt{2})$

   (g) $(\sqrt{7}-\sqrt{6})(\sqrt{7}+\sqrt{6})(\sqrt{x}+1)(\sqrt{x}-1)$ (h) $\sqrt{75}-\frac{3}{4}\sqrt{48}-5\sqrt{12}$

   (i) $\sqrt{2x^2}+5\sqrt{32x^2}-2\sqrt{98x^2}$ (j) $\sqrt{20a^3}+a\sqrt{5a}+\sqrt{80a^3}$

2. Rationalise the denominators and simplify.

   (a) $\frac{2}{\sqrt{5}}$ (b) $\frac{5}{2+\sqrt{3}}$ (c) $\frac{12}{\sqrt{5}-\sqrt{3}}$

   (d) $\frac{\sqrt{2}+1}{2\sqrt{2}-1}$ (e) $\frac{\sqrt{7}+3\sqrt{2}}{\sqrt{7}-\sqrt{2}}$ (f) $\frac{\sqrt{17}-\sqrt{11}}{\sqrt{17}+\sqrt{11}}$

   (g) $\frac{1}{2\sqrt{2}-\sqrt{3}}$ (h) $\frac{\sqrt{6}+1}{3-\sqrt{5}}$

3. Write as a single fraction.

   (a) $\frac{1}{\sqrt{3}+1}+\frac{1}{\sqrt{3}-1}$ (b) $\frac{2}{\sqrt{7}+\sqrt{2}}+\frac{1}{\sqrt{7}-\sqrt{2}}$

   (c) $\frac{1}{3+\sqrt{3}}+\frac{1}{\sqrt{3}-3}+\frac{1}{\sqrt{3}}$ (d)$\frac{7+\sqrt{5}}{7-\sqrt{5}}+\frac{\sqrt{11}-3}{\sqrt{11}+3}$

   (e) $\frac{3+2\sqrt{2}}{(\sqrt{3}-1)^2}$ (f) $\sqrt{\frac{x+1}{x-1}}+\sqrt{\frac{x-1}{x+1}}-\sqrt{\frac{1}{x^2-1}}$

   (g) $\sqrt{\frac{\sqrt[5]{32}+\sqrt{4}}{2^{-2}-2^{-3}}}$

2.4 Exponential Equations

The exponential equation is an equation in which a variable occurs in the exponent.

$3^x = 81$ and $2^{5x-1} = 32$ are some examples of exponential equations.

Some exponential equations may be solved using the fact that, if $x^n = x^m$, then $n = m$ for $x \neq 0$ and $x \neq 1$.

Example 17.

Solve the equation $2^{3x-1} = 32$.

Solution

$$\begin{aligned} 2^{3x-1} &= 32 \\ 2^{3x-1} &= 2^5 \\ 3x - 1 &= 5 \\ 3x &= 6 \\ x &= 2 \end{aligned}$$

Example 18.

Solve the equation $3^{2x+1} \cdot 27^{x-1} = 81$.

Solution

$$\begin{aligned} 3^{2x+1} \cdot 27^{x-1} &= 81 \\ 3^{2x+1} \cdot (3^3)^{x-1} &= 3^4 \\ 3^{2x+1} \cdot 3^{3x-3} &= 3^4 \\ 3^{5x-2} &= 3^4 \\ 5x - 2 &= 4 \\ 5x &= 6 \\ x &= \frac{6}{5} \end{aligned}$$

Example 19.

Solve the equation $2 \cdot 2^{2x} - 7 \cdot 2^{x} - 4 = 0$.
Solution

$$\begin{aligned} 2 \cdot 2^{2x} - 7 \cdot 2^{x} - 4 &= 0 \\ \textit{Let } 2^x = a. & \\ 2a^2 - 7a - 4 &= 0 \\ (2a+1)(a-4) &= 0 \\ 2a + 1 &= 0 \quad \text{or} \quad a - 4 = 0 \\ a &= -\frac{1}{2} \quad \text{or} \quad a = 4 \\ 2^x &= -\frac{1}{2} \quad \text{or} \quad 2^x = 4 \\ \text{impossible} & \quad \text{(or)} \quad 2^x = 2^2 \\ & \qquad \therefore x = 2 \end{aligned}$$

Exercise 2.5

Solve the following equations.

1. $3^{2x-3} = 27^{2x}$
2. $5^{x^2-9} = 1$
3. $5^{x+1} = \frac{1}{625}$
4. $\left(\frac{1}{2}\right)^x = 64$
5. $2^{3x} \cdot 4^{x+1} = 128$
6. $3^{x+1} \cdot 9^{2-x} = \frac{1}{27}$
7. $\frac{27^{2x}}{3^{5-x}} = \frac{3^{2x+1}}{9^{x+3}}$
8. $8^{x-1} = \left(\frac{1}{32}\right)^{x+1}$
9. $10^{-x} = 0.000001$
10. $4^x + 4^{x+1} = 20$
11. $4 \cdot 2^{2x} + 3 \cdot 2^x - 1 = 0$

Chapter 3

Logarithms

In the seventeenth century, a Scottish mathematician John Napier developed method for efficiently performing calculations with large numbers. He found a method for finding the *product* of two numbers by *adding* two corresponding numbers, which he called *logarithms*. The *Richter scale* is a logarithmic scale used to measure the magnitude of an earthquake. pH is another logarithmic scale used to describe how acidic or basic an aqueous solution is.

3.1 Scientific Notation

Scientists often deal with very large or very small numbers. For instance, the speed of the light in vacuum is 299,792,458 metres per second and the mass of an electron is 0.000000000000000000000000000000910938 kilograms. To write such numbers more convenient, scientists make use of a compact form called the scientific notation.

Definition. A positive number is in scientific notation when it is written in the form $a \times 10^n$, where n is an integer and a is a real number satisfying the inequality $1 \leq a < 10$.

In scientific notation, the speed of light mentioned above is approximately 3.0×10^8 metres per second and the mass of an electron is approximately 9.1×10^{-31} kilograms. These numbers are obviously easier to write and remember when given in scientific notation. A simple rule to a write a number in scientific notation is as follow.

Place the decimal point after the first nonzero digit, and produce a new number lying between 1 and 10. Then determine the power of 10 by counting the number of places we moved the original decimal point to the marked decimal point. If we moved the point to the left, then the power is positive; and if we moved it to the right, it is negative.

An approximate number written in scientific notation $a \times 10^n$ indicates an accuracy to the number of digits in "a". The figures or digits in "a" are called **significant figures**. For example, there are three significant figures in the number 3.05×10^{-2}, in which significant digits are 3, 0 and 5.

Example 1.

Write the following numbers in scientific notation.
(a) 14,753 (b) 0.00632 (c) 0.23 (d) 0.00000912 (e) 1,000,000

Solution

(a) $147{,}53 = 1.4753 \times 10^4$
[Decimal point is moved four places to the left.]

(b) $0.00632 = 6.32 \times 10^{-3}$
[Decimal point is moved three places to the right.]

(c) $0.23 = 2.3 \times 10^{-1}$
[Decimal point is moved one place to the right.]

(d) $0.00000912 = 9.12 \times 10^{-6}$
[Decimal point is moved six places to the right.]

(e) $1{,}000{,}000 = 1 \times 10^6$
[Decimal point is moved six places to the left.]

Example 2.

Express the following numbers in ordinary decimal form.
(a) 7.354×10^5 (b) 3.2×10^{-1}

Solution

(a) $7.354 \times 10^5 = 735400$

(b) $3.2 \times 10^{-1} = 0.32$

In carrying out calculations involving measurements, the general rule is that the accuracy of a calculated result is limited by the least accurate measurement involved in the calculation.

1. In addition and subtraction, the result should be rounded so that it has *the same number of digits after the decimal* as the measurement with the least number of digits to the right of the decimal.

   For example, $100.1 + 54.52147 = 154.62147$, which should be rounded as 154.6. Here we can write $100.1 + 54.52147 = 154.62147 \approx 154.6$ or briefly $100.1 + 54.52147 \approx 154.6$. Notice that 100.1 has one digit to the right of the decimal and 54.52147 has five, so the sum must have only one digit after the decimal.

2. In multiplication and division, the result should have *the same number of significant figures* as the measurement with the least.

   For example, $5.01 \times 45.0536 = 225.718536$, which should be rounded as 226. In this case, we can also write $5.01 \times 45.0536 = 225.718536 \approx 226$ or briefly $5.01 \times 45.0536 \approx 226$. Notice that 5.01 ($=5.01 \times 10^0$) has three significant figures and 45.0536 ($=4.50536 \times 10^1$) has six significant figures. So the product must have three significant figures.

Example 3.

Evaluate each of the followings and express the results in scientific notation:

(a) $4.215 \times 10^{-2} + 3.2 \times 10^{-4}$ [Addition]

(b) $8.97 \times 10^4 - 2.62 \times 10^3$ [Subtraction]

(c) $\left(6.73 \times 10^{-5}\right)\left(2.91 \times 10^2\right)$ [Multiplication]

(d) $\dfrac{6.4\times10^6}{1.92\times10^2}$ [Division]

(e) $\left(6.5 \times 10^{-3}\right)^2$ [Power]

(f) $\sqrt{3.6 \times 10^5}$ [*Root*]

Solution

(a) $4.215 \times 10^{-2} + 3.2 \times 10^{-4}$

$= 4.215 \times 10^{-2} + 0.032 \times 10^{-2}$

$= (4.215 + 0.032) \times 10^{-2} \quad = 4.247 \times 10^{-2}$

(b) $8.97 \times 10^4 - 2.62 \times 10^3$

$= 8.97 \times 10^4 - 0.262 \times 10^4$

$= (8.97 - 0.262) \times 10^4 = 8.708 \times 10^4 \approx 8.71 \times 10^4$

(c) $\left(6.73 \times 10^{-5}\right)\left(2.91 \times 10^2\right) = 19.5843 \times 10^{-3} \approx 1.96 \times 10^{-2}$

(d) $\frac{6.4 \times 10^6}{1.92 \times 10^2} \approx 3.3 \times 10^4$

(e) $\left(6.5 \times 10^{-3}\right)^2 = 42.25 \times 10^{-6} \approx 4.2 \times 10^{-5}$

(f) $\sqrt{3.6 \times 10^5} = \sqrt{36 \times 10^4} = 6.0 \times 10^2$

Example 4.

Evaluate $\frac{2,750,000 \times 0.015}{750}$ by transforming each number to scientific notation.

Solution

$$\frac{2,750,000 \times 0.015}{750} = \frac{\left(2.75 \times 10^6\right)\left(1.5 \times 10^{-2}\right)}{7.5 \times 10^2} = 0.55 \times 10^2 = 55$$

Exercise 3.1

1. How many significant figures are there in each of the following numbers?
   (a) 2.175 (b) 0.2175 (c) 0.0075
   (d) 89400 (e) 0.00046

2. Write in scientific notation.
   (a) 24.86 (b) 2.486 (c) 0.2486 (d) 0.002486
   (e) 0.073 (f) 0.0086 (g) 0.934 (h) 7
   (i) 0.00056857 (j) 6.843250

3. Write each number in ordinary decimal form.
   (a) 7.84×10^4 (b) 7.89×10^{-4}
   (c) 2.25×10^5 (d) 4.01×10^{-3}

4. Simplify and give the answers in scientific notation.
   (a) $2.3 \times 10^2 + 1.7 \times 10^2$ (b) $4.6 \times 10^{-3} - 2.5 \times 10^{-3}$
   (c) $(4.5 \times 10^6) \times (1.5 \times 10^{-2})$ (d) $\frac{7.6 \times 10^5}{1.9 \times 10^{-2}}$

5. Compute using scientific notation.

(a) $\frac{2.5\times10^2}{0.25\times0.002}$ (b) $\frac{33{,}000{,}000\times0.4}{1.1\times30}$

(c) $\frac{50\times0.014\times0.30}{10500}$ (d) $\frac{7000\times80\times300}{400}$

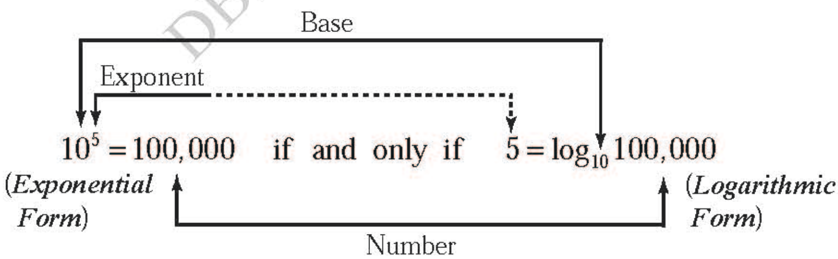
3.2 Definition of the Logarithm

We can easily see that $x = 3$ is the solution of the exponential equation $2^x = 8$. However, for some equations such as $2^x = \frac{1}{3}$ and $2.718^x = 5$, it is not easy to find the solutions. The following basic property of real numbers plays important role in defining logarithms.

"Given a positive number N, and a positive number b other than 1, the equation $N = b^x$ has exactly one solution for x."

Definition. Let N and b be positive real numbers, with $b \neq 1$. Then the **logarithm** of N (with respect) to the base b is the exponent by which b must be raised to yield N, and is denoted by $\log_b N$.

It means that $x = \log_b N$ is the solution of the equation $b^x = N$. In other words, $b^x = N$ if and only if $\log_b N = x$. We say that $b^x = N$ and $\log_b N = x$ are equivalent.



The followings are immediate consequences of the definition of logarithm:

L1.	$N = b^{\log_b N}$
L2.	$x = \log_b b^x$
L3.	$\log_b b = 1$
L4.	$\log_b 1 = 0$

Example 5.

Write each of the following in logarithmic form.

(a) $10^{-2} = 0.01$ (b) $4^{1/2} = 2$ (c) $7^2 = 49$

Solution

(a) $\log_{10} 0.01 = -2$ (b) $\log_4 2 = 1/2$ (c) $\log_7 49 = 2$

Example 6.

Express each of the following in exponential form.

(a) $\log_2 8 = 3$ (b) $\log_{10} 1 = 0$ (c) $\log_5 \left(\frac{1}{\sqrt{5}}\right) = -1/2$

Solution

(a) $2^3 = 8$ (b) $10^0 = 1$ (c) $5^{-1/2} = \frac{1}{\sqrt{5}}$

Example 7.

Find the value of each logarithm.

(a) $\log_5 25$ (b) $\log_2 16\sqrt{2}$ (c) $\log_{1/2} 8$

Solution

(a) Since $5^2 = 25$, we have $\log_5 25 = 2$. (using definition of logarithm)

or $\log_5 25 = \log_5 5^2 = 2$ (using L2)

(b) $\log_2 16\sqrt{2} = \log_2 \left(2^4 \times 2^{1/2}\right) = \log_2 2^{9/2} = \frac{9}{2}$

(c) $\log_{1/2} 8 = \log_{1/2} 2^3 = \log_{1/2} (1/2)^{-3} = -3$

Example 8.

Evaluate each expression.

(a) $3^{\log_3 7} + \log_5 125$ (b) $\log_7 7^9 - \log_3 \frac{1}{9}$

(c) $\log_5 (\log_2 (\log_3 9))$ (d) $10^{2+\log_{10} 5}$

Solution

(a) $3^{\log_3 7} + \log_5 125 = 7 + \log_5 5^3 = 7 + 3 = 10$

(b) $\log_7 7^9 - \log_3 \frac{1}{9} = 9 - \log_3 3^{-2} = 9 - (-2) = 9 + 2 = 11$

(c) $\log_5 (\log_2 (\log_3 9)) = \log_5 (\log_2 (\log_3 3^2)) = \log_5 (\log_2 2) = \log_5 1 = 0$

(d) $10^{2+\log_{10} 5} = 10^2 \times 10^{\log_{10} 5} = 100 \times 5 = 500$

Example 9.

(a) Given that $10^{0.3010} = 2$, find the value of $\log_{10} 16$.

(b) Solve $\log_3 (x^2 - 1) = 2$.

Solution

(a) Since $10^{0.3010} = 2$, we have $\log_{10} 2 = 0.3010$.

So $\log_{10} 16 = \log_{10} 2^4 = 4 \log_{10} 2 = 4\,(0.3010) = 1.204$.

(b) Since $\log_3 (x^2 - 1) = 2$, we have $x^2 - 1 = 3^2$.

$\therefore x^2 = 10$

$\therefore x = \pm\sqrt{10}$

Exercise 3.2

1. Write the following equations in logarithmic form.
   (a) $3^4 = 81$ (b) $9^{3/2} = 27$ (c) $10^{-3} = 0.001$
   (d) $3^{-1} = \frac{1}{3}$ (e) $\left(\frac{1}{4}\right)^{-3} = 64$

2. Write the following equations in exponential form.
   (a) $\log_{10}3 = 0.4771$ (b) $\log_6 0.001 = -3.855$ (c) $\log_{144}12 = \frac{1}{2}$
   (d) $-5 = \log_3 \frac{1}{243}$ (e) $\log_x 7 = y^2$, where $0 < x < 1$

3. Solve the following equations
   (a) $\log_7 49 = x$ (b) $\log_x 10 = 1$ (c) $\log_{\sqrt{3}} x = 2$ (d) $x^{\log_x x} = 5$.

4. Evaluate.
   (a) $9^{\log_9 2} + 3^{\log_3 8}$ (b) $\log_4 4^5 \times \log_{10} 10^2$ (c) $7^{\log_7 9} + \log_2\left(\frac{1}{2}\right)$
   (d) $\log_{\frac{1}{2}} \frac{1}{8} - 4\log_{10}10$ (e) $10^{1-\log_{10}3}$

5. Find the value of x in each of the following problems.
   (a) $\log_3(2x-5) = 2$, where $x > \frac{5}{2}$
   (b) $\log_{77}(\log_7 x) = 0$, where $x > 0$
   (c) $8 + 3^x = 10$, given that $\log_3 2 = 0.6309$

3.3 Properties of Logarithm

The following theorem concerns with some properties of logarithm.

Theorem 1. If M, N, b are positive real numbers, $b \neq 1$ and p is any real number, then

L5. $\log_b(MN) = \log_b M + \log_b N$

L6. $\log_b N^p = p \log_b N$

L7. $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$

Proof.

L5: Since $M = b^{\log_b M}$and $N = b^{\log_b N}$ (using L1), we have
$MN = b^{\log_b M} b^{\log_b N} = b^{\log_b M + \log_b N}$
and hence $\log_b (MN) = \log_b M + \log_b N$.

L6: Since $N = b^{\log_b N}$, we have $N^p = \left(b^{\log_b N}\right)^p = b^{p \log_b N}$
and hence $\log_b N^p = p \log_b N$.

L7: $\log_b \left(\frac{M}{N}\right) = \log_b (MN^{-1})$
$= \log_b M + \log_b N^{-1}$ (using L5)
$= \log_b M - \log_b N$ (using L6)

Basic properties of exponents and logarithms can be summarized as follow.

Properties	**Exponents**	**Logarithms**
One-to-one Property	If $b^x = b^y$, then $x = y$.	If $\log_b M = \log_b N$, then $M = N$.
Product Property	$b^x \cdot b^y = b^{x+y}$	$\log_b (MN) = \log_b M + \log_b N$
Quotient Property	$\frac{b^x}{b^y} = b^{x-y}$	$\log_b \frac{M}{N} = \log_b M - \log_b N$
Power Property	$(b^x)^y = b^{x \cdot y}$	$\log_b N^p = p \log_b N$

Example 10.

If $p = \log_b 2$, $q = \log_b 3$ and $r = \log_b 5$, write $\log_b \frac{5\sqrt{3}}{2}$ in terms of p, q and r.

Solution

$\log_b \frac{5\sqrt{3}}{2} = \log_b 5 + \log_b \sqrt{3} - \log_b 2 = \log_b 5 + \frac{1}{2} \log_b 3 - \log_b 2 = r + \frac{1}{2} q - p$

Example 11.

Using $\log_2 3 = 1.5850$, find the values of

(a) $\log_2 24$ (b) $\log_2 0.75$.

Solution

(a) $\log_2 24 = \log_2 (2^3 \times 3) = \log_2 2^3 + \log_2 3 = 3 + 1.5850 = 4.5850$

(b) $\log_2 0.75 = \log_2 \left(\frac{3}{4}\right) = \log_2 \left(\frac{3}{2^2}\right) = \log_2 3 - \log_2 2^2 = 1.5850 - 2 = -0.4150$

Example 12.

Write each expression as a single logarithm.

(a) $2 + 3\log_5 x^2$ (b) $\log_3 2 + \log_9 81$

(c) $\log_b(3x) + \log_b(4y) - \log_b(2z)$

(d) $-\log_3(2s) + \frac{1}{2}\log_3(4t^2v^4) - 2\log_3(5u)$

Solution

(a) $2 + 3\log_5 x^2 = \log_5 5^2 + \log_5 (x^2)^3 = \log_5 25 + \log_5 x^6 = \log_5(25x^6)$

(b) $\log_3 2 + \log_9 81$

$= \log_3 2 + \log_9 9^2 = \log_3 2 + 2 = \log_3 2 + \log_3 3^2 = \log_3(2 \times 3^2) = \log_3 18$

(c) $\log_b(3x) + \log_b(4y) - \log_b(2z) = \log_b\left(\frac{3x \cdot 4y}{2z}\right) = \log_b\left(\frac{6xy}{z}\right)$

(d) $-\log_3(2s) + \frac{1}{2}\log_3(4t^2v^4) - 2\log_3(5u)$

$= -\log_3(2s) + \log_3(4t^2v^4)^{\frac{1}{2}} - \log_3(5u)^2$

$= \log_3\dfrac{\sqrt{4t^2v^4}}{2s(5u)^2}$

$= \log_3\dfrac{2\,t\,v^2}{2\,s \cdot 25u^2} = \log_3\dfrac{tv^2}{25\,s\,u^2}$

Example 13.

Solve the following equation for x.

$\log_2(3x^2 - 1) - \log_2(2x) = 0$

Solution

$$\begin{aligned}
\log_2\left(3x^2 - 1\right) - \log_2(2x) &= 0 \\
\log_2\left(3x^2 - 1\right) &= \log_2(2x) \\
3x^2 - 1 &= 2x \\
3x^2 - 2x - 1 &= 0 \\
(3x + 1)(x - 1) &= 0 \\
3x + 1 = 0 \text{ or } x - 1 &= 0
\end{aligned}$$

$x = -\frac{1}{3}$ or $x = 1$

But $x = -\dfrac{1}{3}$ is impossible because $\log_2(-\frac{2}{3})$ is not defined.

$\therefore x = 1$

Example 14.

Suppose that $\log_b(xy^2) = 4$ and $\log_b\left(\frac{x^3}{y}\right) = 5$.

(a) Write the equation connecting $\log_b x$ and $\log_b y$.

(b) Find the values of $\log_b x$ and $\log_b y$.

(c) Find $\log_b(y^5\sqrt{x})$.

(d) Write x and y in terms of b.

Solution

(a) Since $\log_b(xy^2) = 4$,

$\log_b x + \log_b y^2 = 4$

$\log_b x + 2\ \log_b y = 4$ (1)

(b) Since $\log_b\left(\frac{x^3}{y}\right) = 5$,

$\log_b x^3 - \log_b y = 5$

$3\ \log_b x - \log_b y = 5$

$6\ \log_b x - 2\log_b y = 10$ (2)

Adding equations (1) and (2),

$7\ \log_b x = 14$

$\log_b x = 2$

Substituting $\log_b x = 2$ in (1),

$2 + 2\ \log_b y = 4$

$\log_b y = 1$

(c) $\log_b(y^5\sqrt{x}) = \log_b y^5 + \log_b\sqrt{x} = 5\log_b y + \frac{1}{2}\log_b x = 5\,(1) + \frac{1}{2}\,(2) = 6$

(d) Since $\log_b y = 1$, we have $y = b$.

Since $\log_b x = 2$, we have $x = b^2$.

Exercise 3.3

1. Replace $\square$ with the appropriate number.
   (a) $\log_3 24 = \log_3 6 + \log_3 \square$ (b) $\log_5 24 = \log_5 60 + \log_5 \square$
   (c) $\log_2 \square = 3\log_2 3$ (d) $\log_{10} 9 = \square \log_{10} 3$
   (e) $\log_8 5 = \log_8 \square - \log_8 11$
2. Write each expression as a single logarithm.
   (a) $\log_b 20 + \log_b 57 - \log_b 114$ (b) $3\log_b 8 - \log_b 12$
   (c) $\log_b x - 2\log_b y - \log_b a$ (d) $\log_2 3 + \log_4 15$
3. Write each expression in terms of $\log_b 2$, $\log_b 3$ and $\log_b 5$.
   (a) $\log_b 8$ (b) $\log_b 15$ (c) $\log_b 270$
   (d) $\log_b \frac{27\sqrt[3]{5}}{16}$ (e) $\log_b \frac{216}{\sqrt[3]{32}}$ (f) $\log_b \left(648\sqrt{125}\right)$
4. Evaluate each expression.
   (a) $\log_2 128$ (b) $\log_3 81^4$ (c) $\log_{1/2} 8$
   (d) $\log_8 2$ (e) $\log_3 \frac{\sqrt{3}}{81}$ (f) $\frac{\log_3 \sqrt{3}}{\log_3 81}$
   (g) $\frac{\log_2 25}{\log_2 5}$ (h) $\log_4 8$
5. Use $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.4771$ to evaluate each of the following expressions.
   (a) $\log_{10} 6$ (b) $\log_{10} 1.5$ (c) $\log_{10}\sqrt{3}$
   (d) $\log_{10} 4$ (e) $\log_{10} 4.5$ (f) $\log_{10} 8$
   (g) $\log_{10} 18$ (h) $\log_{10} 5$
6. Solve the following equations for x.
   (a) $\log_a \frac{18}{5} + \log_a \frac{10}{3} - \log_a \frac{6}{7} = \log_a x$
   (b) $\log_b x = 2 - a + \log_b \left(\frac{a^2 b^a}{b^2}\right)$
   (c) $\log_b x^3 - \log_b x^2 = \log_b 5x - \log_b 4x$
   (d) $\log_{10} x + \log_{10} 3 = \log_{10} 6$
   (e) $8\log_b x = \log_b a^{\frac{3}{2}} + \log_b 2 - \frac{1}{2}\log_b a^3 - \log_b \frac{2}{a^4}$

7. Given that $\log_{10} 5 = 0.6990$ and $\log_{10} x = 0.2330$. What is the value of x ?

8. Show that if $\log_e I = -\frac{R}{L}t + \log_e I_0$ then $I = I_0 e^{-\frac{Rt}{L}}$.

9. Show that if $\log_b y = \frac{1}{2}\log_b x + c$ then $y = b^c\sqrt{x}$.

10. Show that
    (a) $\frac{1}{4}\log_{10} 8 + \frac{1}{4}\log_{10} 2 = \log_{10} 2$
    (b) $4\log_{10} 3 - 2\log_{10} 3 + 1 = \log_{10} 90$.

11. Show that
    (a) $a^{2\log_a 3} + b^{3\log_b 2} = 17$
    (b) $3\ \log_6 1296 = 2\ \log_4 4096$.

12. Given that $\log_{10} 12 = 1.0792$ and $\log_{10} 24 = 1.3802$, deduce the values of $\log_{10} 2$ and $\log_{10} 6$.

13. If $\log_x a = 5$ and $\log_x 3a = 9$, find the values of a and x.

14. (a) If $\log_{10} 2 = a$, find $\log_{10} 8 + \log_{10} 25$ in terms of a.
    (b) If $a = 10^x$ and $b = 10^y$, express $\log_{10}(a^4 b^3)$ in terms of x and y .

15. (a) If $\log_2(4x - 4) = 2$, find the value of $\log_4 x$.
    (b) Prove that if $\frac{1}{2}\log_3 M + 3\log_3 N = 1$ then $MN^6 = 9$.

3.4 Change of Base

Logarithms to any base can be computed by using logarithms to some other bases.

Theorem 2. Suppose a and b are any two positive real numbers other than 1. If N is any positive real number, then

L8. $\log_a N = \frac{\log_b N}{\log_b a}$

Proof.

Since $N = a^{\log_a N}$, we have

$\log_b N = \log_b a^{\log_a N} = (\log_a N)(\log_b a)$.

So $\log_a N = \frac{\log_b N}{\log_b a}$

Corollary 1. If a and N are any two positive real numbers other than 1, then

L9. $\log_a N = \frac{1}{\log_N a}$

Proof.

Using *L8* and *L3*, $\log_a N = \frac{\log_N N}{\log_N a} = \frac{1}{\log_N a}$.

Corollary 2. If a, p, N are any positive real numbers such that $a \neq 1$, then

L10. $\log_{a^p} N = \frac{1}{p} \log_a N$

Proof.

L10 follows from *L9* and *L6*.

Corollary 3. If a, b, k are any positive real numbers other than 1, then

L11. $a^{\log_k b} = b^{\log_k a}$

Proof.

Using L2 and L6, $a^{\log_k b} = a^{\log_k\left(a^{\log_a b}\right)} = a^{(\log_a b)(\log_k a)} = \left(a^{\log_a b}\right)^{\log_k a} = b^{\log_k a}$

Example 15.

Given that $\log_p x = 20$ and $\log_p y = 5$, find $\log_y x$ and $\log_x y$.
Solution
$\log_y x = \frac{\log_p x}{\log_p y} = \frac{20}{5} = 4$
$\log_x y = \frac{1}{\log_y x} = \frac{1}{4}$

Example 16.

Find the value of
(a) $2^{\frac{\log_5 3}{\log_5 2}}$ (b) $5^{\frac{1}{\log_7 5}}$ (c) $\log_3 5 \times \log_{25} 27$.
Solution
(a) $2^{\frac{\log_5 3}{\log_5 2}} = 2^{\log_2 3} = 3$
(b) $5^{\frac{1}{\log_7 5}} = 5^{\log_5 7} = 7$
(c) $\log_3 5 \times \log_{25} 27 = \log_3 5 \times \log_{5^2} 3^3$
$= \log_3 5 \times \frac{3}{2} \log_5 3$
$= \log_3 5 \times \frac{3}{2} \times \frac{1}{\log_3 5} = \frac{3}{2}$

Example 17.

Solve the equation $\log_3 x = 3 - 2\log_x 3$, where $x > 0$ and $x \neq 1$.
Solution

$$\begin{aligned}
\log_3 x &= 3 - 2\log_x 3 \\
\log_3 x &= 3 - \frac{2}{\log_3 x} \\
(\log_3 x)^2 &= 3\log_3 x - 2 \\
(\log_3 x)^2 - 3\log_3 x + 2 &= 0 \\
(\log_3 x - 2)(\log_3 x - 1) &= 0 \\
\log_3 x - 2 = 0 \quad &\text{or} \quad \log_3 x - 1 = 0 \\
\log_3 x = 2 \quad &\text{or} \quad \log_3 x = 1 \\
x = 3^2 \quad &\text{or} \quad x = 3 \\
x = 9 \quad &\text{or} \quad x = 3
\end{aligned}$$

Example 18.

Solve the logarithmic equation $\log_3 x = \log_9 (x+6)$.
Solution

$$\begin{aligned}
\log_3 x &= \log_9 (x+6) \\
\log_3 x &= \log_{3^2} (x+6) \\
\log_3 x &= \frac{1}{2}\log_3 (x+6) \\
2\log_3 x &= \log_3 (x+6) \\
\log_3 x^2 &= \log_3 (x+6) \\
x^2 &= x+6 \\
x^2 - x - 6 &= 0 \\
(x-3)(x+2) &= 0 \\
x = 3 \quad \text{or} \quad & x = -2
\end{aligned}$$

But $x = -2$ is impossible since $\log_3(-2)$ is undefined.
So $x = 3$.

Exercise 3.4

1. If $\log_a b + \log_b a^2 = 3$, find b in terms of a.

2. Show that
   (a) $\log_4 x = 2\log_{16} x$ (b) $\log_b x = 3\log_{b^3} x$ (c) $\log_2 x = (1 + \log_2 3)\log_6 x$.

3. If $a = \log_b c$, $b = \log_c a$ and $c = \log_a b$, prove that $abc = 1$.

4. Show that
   (a) $(\log_{10} 4 - \log_{10} 2)\log_2 10 = 1$ (b) $2\log_2 3(\log_9 2 + \log_9 4) = 3$.

5. Compute
   (a) $3^{\log_2 5} - 5^{\log_2 3}$ (b) $4^{\log_2 3}$ (c) $2^{\log_{2\sqrt{2}} 27}$.

3.5 Common Logarithm and Natural Logarithm

The logarithm of N to the base 10 is said to be a **common logarithm**, and is usually written as $\log N$. If $N = a \times 10^n$, then

$$\log N = \log(a \times 10^n) = \log 10^n + \log a = n + \log a;$$

in this case n and $\log a$ are respectively said to be the **characteristic** and the **mantissa** of $\log N$.

As a positive integer n become very large, the value of $(1 + 1/n)^n$ approaches an irrational number, which is denoted by e. The number e is called **Euler's number**, in honour of the Swiss mathematician Leonard Euler, and is approximately equal to 2.71828. The logarithm of N to the base e is called a *natural logarithm*, and is denoted by $\ln N$.

Example 19.
Given that $\log_{10}7 = 0.8451$; what are the characteristics and the mantissas of $\log_{10}0.007$ and $\log_{10}700$?

Solution

$\log_{10}7 = 0.8451$

$\log_{10}0.007 = \log_{10}(7 \times 10^{-3}) = \log_{10}10^{-3} + \log_{10}7 = -3 + 0.8451$

The characteristic and the mantissa of $\log_{10}0.007$ are -3 and 0.8451 respectively.

$\log_{10}700 = \log_{10}(7 \times 10^2) = \log_{10}10^2 + \log_{10}7 = 2 + 0.8451$

The characteristic and the mantissa of $\log_{10}700$ are 2 and 0.8451 respectively.

Notice that the mantissas of $\log_{10}7$, $\log_{10}0.007$ *and* $\log_{10}700$ *are the same. Characteristics can be seen directly from the scientific notation of the numbers.*

Example 20.

Two nonnegative real numbers A and P are related by the formula $A = Pe^{0.085t}$. Given that $\ln 2 = 0.6931$, find the value of t for which A becomes 200% of P.

Solution

We have $A = Pe^{0.085t}$. If $A = 200\%$ of P, then

$\frac{200}{100} \times P = Pe^{0.085t}$.
So $2 = e^{0.085t}$

$\log_e 2 = 0.085\, t$
$$t = \frac{\ln\ 2}{0.085} = \frac{0.6931}{0.0850} = \frac{6931}{850} = 8\frac{131}{850}$$

Example 21.

Using $\log_{10}2 = 0.3010$, $\log_{10}9.87 = 0.9943$ and $\log_{10}8.5 = 0.9294$; evaluate $\dfrac{200 \times 98.7 \times 85}{8.5^3}$.

Solution

Let $p = \dfrac{200 \times 98.7 \times 85}{8.5^3}$. Then

$$\begin{aligned}
\log_{10}p &= \log_{10}\frac{200 \times 98.7 \times 85}{8.5^3} \\
&= \log_{10}200 + \log_{10}98.7 + \log_{10}85 - \log_{10}8.5^3 \\
&= \log_{10}\left(2 \times 10^2\right) + \log_{10}\left(9.87 \times 10^1\right) + \log_{10}(8.5 \times 10) - 3\log_{10}8.5 \\
&= 2 + \log_{10}2 + 1 + \log_{10}9.87 + 1 + \log_{10}8.5 - 3\log_{10}8.5 \\
&= 4 + \log_{10}2 + \log_{10}9.87 - 2\log_{10}8.5 \\
&= 4 + 0.3010 + 0.9943 - 2 \times 0.9294 \\
&= 5.2953 - 1.8588 \\
&= 3.4365
\end{aligned}$$

So $\dfrac{200 \times 98.7 \times 85}{8.5^3} = p = 10^{3.4365}$

Exercise 3.5

1. Given that $\log 2.345 = 0.3701$. What are the characteristics and the mantissas of each of the followings?
   (a) $\log 234,500$ (b) $\log 0.0002345$

2. Using $\log_{10}2.74 = 0.4378$, $\log_{10}2.83 = 0.4518$, $\log_{10}5.97 = 0.7760$, $\log_{10}6.21 = 0.7931$, $\log_{10}8.18 = 0.9128$ and $\log_{10}9.27 = 0.9671$, compute
   (a) $\left(\dfrac{28.3}{597 \times 621}\right)^2$ (b) $\dfrac{274^{\frac{1}{3}}}{927 \times 818}$ (c) $\dfrac{28.3\sqrt{0.621}}{597}$.

Chapter 4

Functions

In this chapter, product sets, relations and functions are introduced. A relation is described as a set of ordered pairs, and a function is described as a special kind of relation. Then some basic functions are illustrated by graphs. You will identify the relation, function, domain and range, and describe each of these mathematical concept in a given context. You will define the equality of functions, one-to-one function, inverse function and composite functions and solve with these functions in algebraic form.

4.1 Product Sets

Let a and b be any two elements. When we indicate such two elements a and b as an ordered, we take a as the first element and b as the second element and enclose the elements in parentheses (a, b). A pair of such elements a and b, written as (a, b), is called **an ordered pair**.

Two ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.

Definition. Let A and B be any two sets. The product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. In symbol

$$A \times B = \{(a, b) \mid a \in A \quad \text{and} \quad b \in B\}.$$

$A \times B$ is read as A **cross** B.

Example 1.

Let $A = \{2, 4\}$, $\quad B = \{1, 5, 6\}$. Find $A \times B$ and $B \times A$.

Solution

$A \times B = \{(2,1), (2,5), (2,6), (4,1), (4,5), (4,6)\}$.

$\times$	1	5	6
2	(2, 1)	(2, 5)	(2, 6)
4	(4, 1)	(4, 5)	(4, 6)

$B \times A = \{(1,2), (1,4), (5,2), (5,4), (6,2), (6,4)\}$.

$\times$	2	4
1	(1, 2)	(1, 4)
5	(5, 2)	(5, 4)
6	(6, 2)	(6, 4)

Similarly, we can write

$$\begin{aligned} A \times A &= \{(2,2), (2,4), (4,2), (4,4)\}, \\ B \times B &= \{(1,1), (1,5), (1,6), (5,1), (5,5), (5,6), (6,1), (6,5), (6,6)\}. \end{aligned}$$

Here the order of elements in the ordered pair is important since $(2, 1)$ and $(1, 2)$ are two different ordered pairs.

We can count the number of elements of $A \times B$. In example 1, A has 2 elements, B has 3 elements, $A \times B$ has 6 elements and $A \times A$ has 4 elements. Therefore,

the number of elements of $A \times B = 6 = 2 \times 3$,

and the number of elements of $A \times A = 4 = 2 \times 2$.

In general, if A has m elements and B has n elements, then $A \times B$ has mn elements.

Therefore, the number of elements of $A \times B = mn$.

Exercise 4.1

1. Let $S = \{1, 2, 3, 4, 5, 6, 7\}$ be a universal set, $A = \{x \mid x \text{ is a prime number}\}$ and $B = \{x \mid 1 < x < 5\}$. Find $A \times B$ and $B \times A$.
2. If $A = \{1, 2\}$, $B = \{2, 3, 4\}$, find $(B \setminus A) \times (A \cup B)$.
3. Compute $\{a\} \times \{a\}$.

4. Let $A = \{1, 2\}$, $B = \{1, 2, 3, 4\}$ and $C = \{a, b\}$. Show that $A \times C \subset B \times C$.

5. $A = \{1, 2, 3\}$, $B = \{3, 4\}$, $C = \{4\}$, $D = \{1, 2\}$. Prove the following:

   (a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

   (b) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

   (c) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

4.2 Relations

Definition. A **relation** is a set of ordered pairs.

For example, the set $R = \{(2, 2), (2, 4), (6, 3)\}$ is a relation.

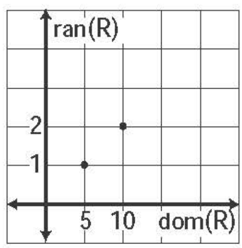
Definition. If R is a relation, the set of all first elements x of ordered pairs $(x, y) \in R$ is called the **domain** of R, denoted by dom(R). The set of all second elements y is called the **range** of R, denoted by ran(R).

For example, in the relation $R = \{(2, 2), (2, 4), (6, 3)\}$ has dom$(R) = \{2, 6\}$ and ran$(R) = \{2, 3, 4\}$.

Graph of a Relation

We can draw a graph which describes a relation as follows.

Consider the relation $R = \{(5, 1), (10, 2)\}$. We have that dom$(R) = \{5, 10\}$ and ran$(R) = \{1, 2\}$. Place the elements of dom(R) on a horizontal line and the elements of ran(R) on a vertical line. The dots represent the graph of R.



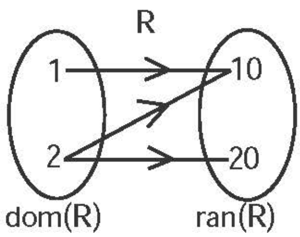
We can draw an arrow diagram of a relation as in the following example.

Example 2.

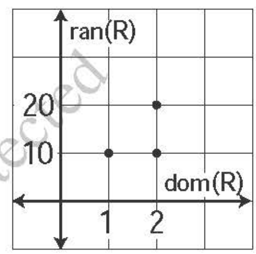
Let $R = \{(1,10),(2,10),(2,20)\}$ be a relation. State the domain and range of R. Draw an arrow diagram and a graph of the relation.

Solution

$R = \{(1,10),(2,10),(2,20)\}$. Domain of R is $\{1,2\}$ and range of R is $\{10,20\}$.



Arrow diagram of R

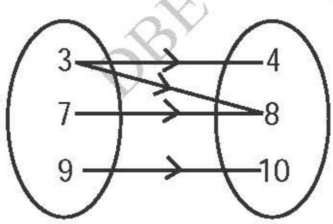


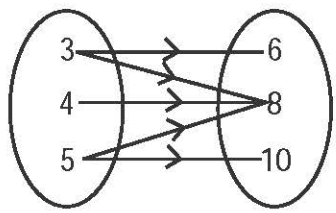
Graph of R

Exercise 4.2

1. Let $R = \{(1,2),(2,4),(2,5),(3,6),(4,8)\}$ be a relation. State the domain and range of R. Draw a graph and an arrow diagram to describe R.

2. Write the sets of ordered pairs that represent the relations for each of the following arrow diagrams. Draw the graph of each relation.





4.3 Functions

Definition. A **function** f is a relation such that $(x,y) \in f$ and $(x,z) \in f$ implies $y = z$. The unique element y such that $(x,y) \in f$ is the **image** of x under f; we use the notation

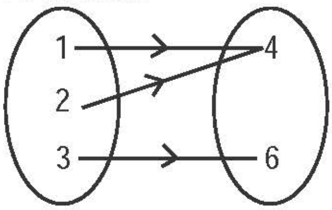
$$y = f(x) \quad \text{(read } f \text{ of } x\text{)} \quad \text{(or)} \quad f : x \mapsto y \ \text{ for } \ (x,y) \in f.$$

The domain and range of the function f are given by

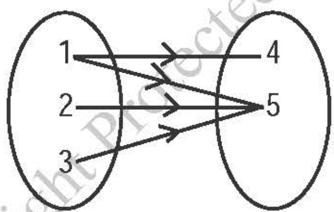
$$\text{dom }(f) = \{x \mid (x,y) \in f\} \text{ and ran } (f) = \{y \mid (x,y) \in f\}.$$

> Note that for every x in the domain of a function f there is exactly one y such that $(x,y) \in f$.

For example, the relation $\{(1,4),(2,4),(3,6)\}$ is a function. The relation $(1,4),(1,5),(2,5),(3,5)$ is not a function. See below the arrow diagrams of these relations.

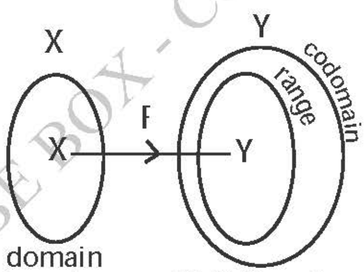


This relation is a function.



This relation is not a function.

Definition.



(i) We say that f is a function **on** X if $X = \text{dom}(f)$.

(ii) We call f is a function **from** X to Y (or a **mapping from** X **into** Y),

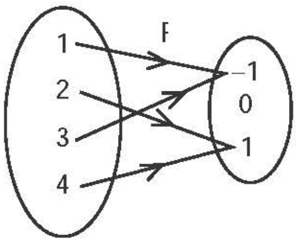
$$f : X \to Y,$$

if $\text{dom}(f) = X$ and $\text{ran}(f) \subset Y$. In this case Y is called the **codomain** of f.

(iii) If $Y = \text{ran}(f)$, then f is a function **onto** Y.

(iv) If f is a function from X to Y and $A \subset X$, then $\{f(a) \mid a \in A\}$ is denoted by $f(A)$ and is called the image of A.

(v) If $B \subset Y$ then $\{x \in X \mid f(x) \in B\}$ is denoted by $f^{-1}(B)$ and is called the inverse image of B.



For example, the given arrow diagram represents the function

$$f = \{(1, -1), (2, 1), (3, -1), (4, 1)\}$$

from the set $\{1, 2, 3, 4\}$ to $\{-1, 0, 1\}$ with the domain $\{1, 2, 3, 4\}$, the range $\{-1, 1\}$ and the codomain $\{-1, 0, 1\}$.

Let $A = \{1, 3\} \subset X$ and $B = \{1\} \subset Y$. Then $f(A) = \{-1\}$ and $f^{-1}(B) = \{2, 4\}$.

Example 3.

Let $A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 6\}$. The relation from A to B is described by (a) $\{(1, 4), (2, 5), (3, 6), (4, 5)\}$, (b) $\{(1, 4), (2, 4), (3, 4), (1, 6)\}$.
Which relation is a function from A to B?

Solution

(a) The relation is a function from A to B because each element of A is related to exactly one element of B.

(b) The relation is not a function because the element 1 of A is related to two elements of B, namely 4 and 6.

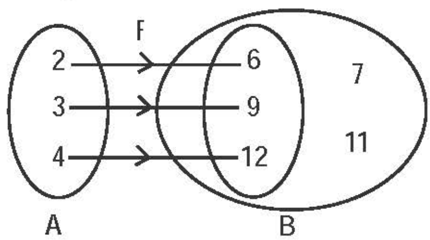
Example 4.

Let $A = \{2, 3, 4\}$ and $B = \{6, 7, 9, 11, 12\}$. Let f be a function from A to B such that $x \mapsto 3x$ where $x \in A$. Find the range of f.

Solution

The domain $A = \{2, 3, 4\}$. Let us find the range.

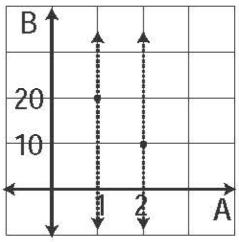
$$2 \mapsto 6$$
$$3 \mapsto 9$$
$$4 \mapsto 12$$



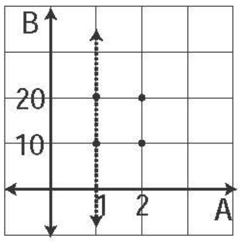
The range of f is $\{6, 9, 12\}$. It is clear that $\text{ran}(f) \subset B$. Thus f is a function from A into B.

Vertical Line Test: A relation is a function if each vertical line intersects the graph at most one point.

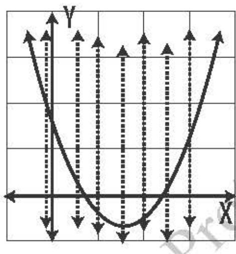
We can use the vertical line test to determine whether the following graphs represent functions or not.



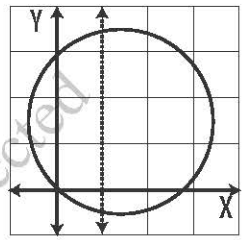
A function



Not a function



A function



Not a function

Example 5.

Let $\mathbb{R}$ be the set of all real numbers. Let the function f be defined by $f(x) = x^2 + x + 3$ for every $x \in \mathbb{R}$. Evaluate (a) $f(2)$ (b) $f(-3)$ (c) $f(-x)$.

Solution

$f(x) = x^2 + x + 3$.

(a) $f(2) = 2^2 + 2 + 3 = 9$.

(b) $f(-3) = (-3)^2 + (-3) + 3 = 9$.

(c) $f(-x) = (-x)^2 + (-x) + 3 = x^2 - x + 3$.

Example 6.

Let the function $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = px + q$, where p and q are real numbers. If $f(1) = 4$ and $f(-2) = 1$, find p and q.

Solution

$f(x) = px + q$.

Since $f(1) = 4$, $p(1) + q = 4$ gives

$$p + q = 4. \qquad (1)$$

Since $f(-2) = 1$, $p(-2) + q = 1$ gives

$$-2p + q = 1. \qquad (2)$$

Solving (1) and (2), we get $p = 1$ and $q = 3$.

The Domain of a Function

> When the domain of a function is not specified, then assume that it is the set of all possible real numbers for which the function makes sense.

Example 7.

State the domain of

(a) $f(x) = \dfrac{1}{x-1}$ (b) $h(x) = \dfrac{x-3}{2}$ (c) $f(x) = \dfrac{1}{x^2-4}$.

Solution

(a) $f(x) = \dfrac{1}{x-1}$ makes sense when $x \neq 1$.

$\therefore$ domain of $f = \{x \mid x \neq 1, x \in \mathbb{R}\}$.

(b) $h(x) = \dfrac{x-3}{2}$ makes sense for all real numbers.

$\therefore$ domain of $h = \{x \mid x \in \mathbb{R}\}$.

(c) $f(x) = \dfrac{1}{x^2-4}$ make sense when $x^2 - 4 \neq 0$. i.e., $x \neq 2$ and $x \neq -2$.

$\therefore$ domain of $f = \{x \mid x \neq 2 \quad \text{and} \quad x \neq -2,\ x \in \mathbb{R}\}$.

Equality of Functions

Two functions f and g are equal (and we write $f = g$) if and only if

1. f and g have the same domain, and
2. $f(x) = g(x)$ for each element x in the domain.

Example 8.

Determine whether $f(x) = x - 2$ and $g(x) = \dfrac{x^2-4}{x+2}$ are equal functions or not.

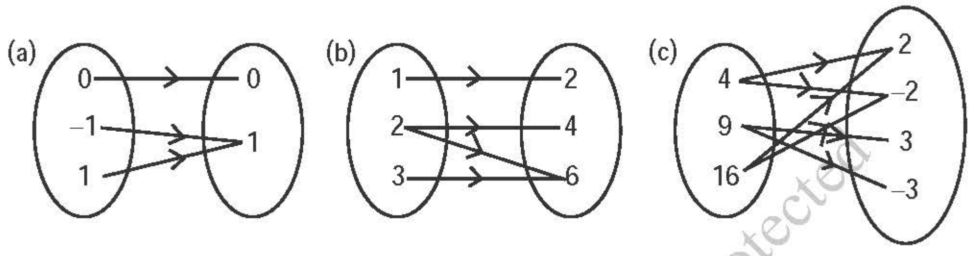
Solution

$\text{dom}(f) = \mathbb{R}$ and $\text{dom}(g) = \{x \mid x \neq -2,\ x \in \mathbb{R}\}$.

The domain of f and g are not the same. Therefore $f \neq g$.

Exercise 4.3

1. Determine whether each relation is a function or not. If it is a function, state the domain and range.



2. Consider the following relations. Determine whether each relation is a function or not. If it is a function, write down the domain and range.
   (a) $\{(1,3),(2,5),(3,7),(4,9)\}$
   (b) $\{(-2,5),(-1,3),(0,1),(-1,1)\}$
   (c) $\{(1,3),(2,3),(3,2),(2,1)\}$
   (d) $\{(2,4),(3,6),(4,6),(7,14)\}$
   (e) $\{(0,0),(1,1),(3,3),(4,4)\}$
   (f) $\{(2,a),(4,c),(5,a),(4,e)\}$

3. Let f be a function from $\mathbb{R} \to \mathbb{R}$. Which of the following statements are true?
   (a) If $f(x) = 5 - x$, the image of -3 under f is 8.
   (b) If $f(x) = x^2 + 9$, the image of -3 under f is zero.
   (c) If $f(x) = 3x + 4$, then $f(a) = a$ implies that $a = -2$.
   (d) If $f(x) = x + 3$, there is only one value $a \in \mathbb{R}$ such that $f(a) = 0$.
   (e) If $f(x) = x^2 - 1$, then there are exactly two values $a \in \mathbb{R}$ such that $f(a) = 0$.

4. Illustrate the function $f : x \mapsto x + 2$ with an arrow diagram for the domain $\{3, 5, 7, 9, 10\}$. Write down the range of f.

5. Let the domain of function $h : x \mapsto 0$ be $\{2, 4, 6, 7\}$. What is the range of h? Draw an arrow diagram for h.

6. Let the domain of function $f : x \mapsto 3x$ be the set of natural numbers less than 5. State the domain and range.

7. Let $g : \mathbb{R} \to \mathbb{R}$ be given by
   (a) $g(x) = 3 - 4x$. Find $g(1)$, $g(3)$, $g(-2)$, $g(x+3)$, $g(\frac{1}{2})$.

(b) $g(x) = 2x - 5$. Find $g(3),\ g(\frac{1}{2}),\ g(0),\ g(-4),\ g(4)$.
If $g(a) = 99$, find a.
(c) $g(x) = \dfrac{x+5}{2}$. Find the images of $3,\ 0,\ -3$. Find x if $g(x) = 0$.
(d) $g(x) = 3x - 1$. Find x such that $g(x) = 20$.
(e) $g(x) = 3x + 1$. Find x such that $g(x) = 22$.

8. A function f from A to A, where A is the set of positive integers, is given by $f(x) =$ the sum of all possible divisors of x.
For example $f(6) = 1 + 2 + 3 + 6 = 12$.
(a) Find the values of $f(2)$, $f(5)$, $f(13)$, $f(18)$.
(b) Show that $f(14) = f(15)$ and $f(3) \cdot f(5) = f(15)$.

9. Let $A =$ the set of positive integers greater than 3 and $B =$ the set of all positive integers. Let $d : A \to B$ be a function given by $d(n) = \frac{1}{2}n(n-3)$, the number of diagonals of a polygon of n sides.
(a) Find $d(6)$, $d(8)$, $d(10)$, $d(12)$.
(b) How many diagonals will a polygon of 20 sides have?

10. Determine whether f and g are equal functions or not. Give reason:
(a) $f(x) = x^2 + 2, \quad g(x) = (x+2)^2$ (b) $f(x) = \dfrac{x^2 - 1}{x + 1}, \quad g(x) = x - 1$
(c) $f(x) = x^2, \quad g(x) = |x|^2$ (d) $f(x) = \dfrac{x + 2}{x^2 - 4}, \quad g(x) = \dfrac{1}{x - 2}$

11. State the domain of the following functions.
(a) $f(x) = \sqrt{x - 2}$ (b) $f(x) = \dfrac{1}{2x - 1}$
(c) $f(x) = \dfrac{4}{x - 3}$ (d) $f(x) = \dfrac{2}{x^2 - 1}$

4.3.1 The Graph of a Function

Let f be any function. The graph of f is the set of ordered pairs given by the function f as

$$\{(x, y) \mid x \in \text{dom}(f) \text{ and } y = f(x)\}.$$

The graph of function f is illustrated by plotting the set of ordered pairs in xy-plane.

We introduce the following curves of basic functions f given by $y = f(x)$.

Linear function: $y = mx + c$ where m and c are constants.

Quadratic function: $y = ax^2$ where a is a constant.

Absolute value function: $y = |x|$.

Square root function: $y = \sqrt{x}$.

Rational function: $y = \dfrac{ax+b}{cx+d}$ where a, b, c and d are constants.

To draw the graph of a function

1. Make a suitable xy-table for the given function.
2. In the xy- plane, plot the points obtained from the xy- table.
3. Draw a curve through the points.

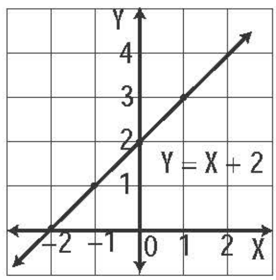
The Graph of Linear Function $y = mx + c$

Example 9.

Graph of $y = x + 2$ $(x \in \mathbb{R})$.

x	...	-2	-1	0	1	...
y	...	0	1	2	3	...

When $y = 0$, $x = -2$.
The graph cuts the x-axis at $(-2, 0)$
When $x = 0$, $y = 2$.
The graph cuts the y-axis at $(0, 2)$.



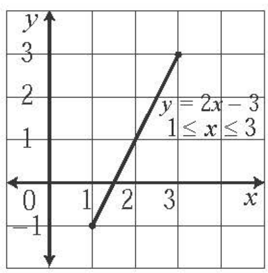
From the figure,
Domain $= \mathbb{R}$.
Range $= \mathbb{R}$.

Example 10.

Graph of $y = 2x - 3$ $(1 \le x \le 3)$.

x	1	...	2	...	3
y	-1	...	1	...	3

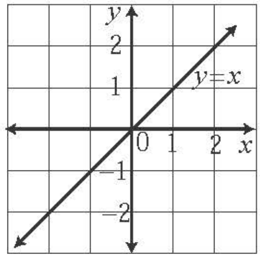
Domain $= \{x \mid 1 \le x \le 3\}$.
Range $= \{y \mid -1 \le y \le 3\}$.



Identity Function. The identity function I on $\mathbb{R}$ is the function $I : \mathbb{R} \to \mathbb{R}$ defined by

$$y = I(x) = x$$

i.e., the image of every $x \in \mathbb{R}$ is just itself.



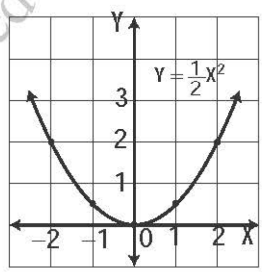
The Graph of Quadratic Function $y = ax^2$ $(a \neq 0)$

Example 11.

Graph of $y = \frac{1}{2}x^2$.

x	...	-2	-1	0	1	2	...
y	...	2	$\frac{1}{2}$	0	$\frac{1}{2}$	2	...

Domain $= \mathbb{R}$.
Range $=\{y \mid y \geq 0\}$.

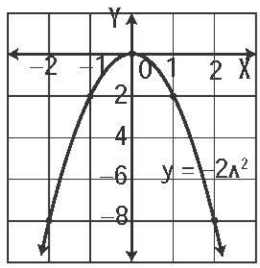


Example 12.

Graph of $y = -2x^2$.

x	...	-2	-1	0	1	2	...
y	...	-8	-2	0	-2	-8	...

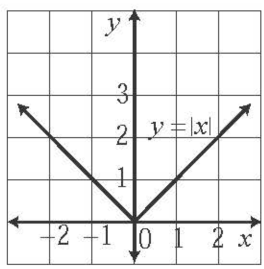
Domain $= \mathbb{R}$.
Range $= \{y \mid y \leq 0\}$.



The Graph of Modulus Function $y = |x|$
If x is a real number, we define

$$|x| = \begin{cases} -x & \text{if} \quad x < 0, \\ 0 & \text{if} \quad x = 0, \\ x & \text{if} \quad x > 0. \end{cases}$$

$|x|$ is called the **modulus**, or the **absolute value** of x.



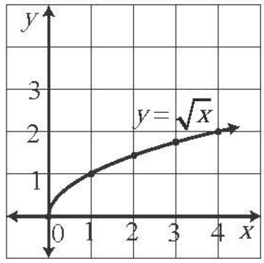
The function $f : \mathbb{R} \to [0, \infty)$ defined by $f(x) = |x|$ is known as the modulus function.

The Graph of Square Root Function $y = \sqrt{x}$

The function f, defined by $f(x) = \sqrt{x}$, is known as the square root function.

Domain $= \{x \mid x \geq 0\}$.

Range $= \{y \mid y \geq 0\}$.



Exercise 4.4

1. Sketch the graphs of:
   (a) $y = x - 1$ (b) $y = -x - 2$ (c) $y = -x + 2$
   (d) $y = 2x + 1$ (e) $y = 3x^2$ (f) $y = -3x^2$
   (g) $y = \frac{1}{3}x^2$ (h) $y = \sqrt{2x}$ (i) $y = |2x|$

2. Sketch the graphs of $y = 2x$ and $y = \frac{1}{2}x$ in the same plane. What do you notice from the graphs? Explain.

3. Sketch the graphs of $y = \frac{1}{2}x^2$ and $y = 2x^2$ in the same plane. What do you notice from the graphs? Explain.

(i) The Graph of $y = \dfrac{k}{x}$, $(k \neq 0)$

To draw a graph of

$$y = \frac{k}{x} \qquad \text{for} \quad k \neq 0,$$

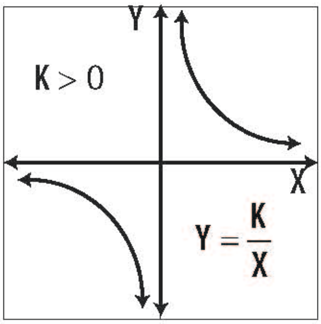
we consider two cases, $k > 0$ and $k < 0$.

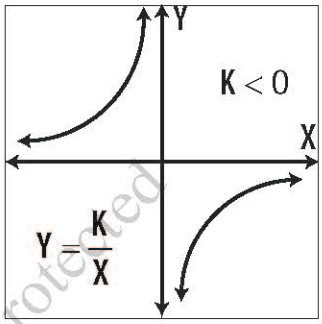
If $k > 0$, both x and y are positive, or negative. Therefore the point (x, y) lies in the quadrant I and quadrant III.

The values of y approach to 0 (zero) while the absolute values of x get larger, and the absolute values of y get larger while the values of x approach 0 (zero). However the curves never intersect both the lines $x = 0$ and $y = 0$. These lines are called the asymptotes. The line $x = 0$ is the **vertical asymptote** and the line $y = 0$ is the **horizontal asymptote**.

If $k < 0$, then $x < 0$ gives $y > 0$ and $x > 0$ gives $y < 0$. Therefore the point (x, y) lies in the quadrant II and quadrant IV.

Therefore we can conclude that if $k > 0$, the curves are in quadrant I and III and if $k < 0$, the curves are in quadrant II and IV.



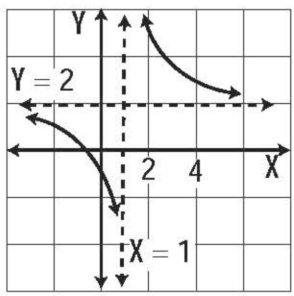


Domain $= \{x \mid x \neq 0,\ x \in \mathbb{R}\}$.
Range $= \{y \mid y \neq 0,\ y \in \mathbb{R}\}$.

(ii) The Graph of $y = \dfrac{k}{x-p} + q$

This equation is defined for $x \neq p$. The line $x = p$ is a vertical asymptote. When the absolute value of x gets larger and larger, y approaches to q. But the curves never intersect the line $y = q$. This line is a horizontal asymptote.

Example 13.



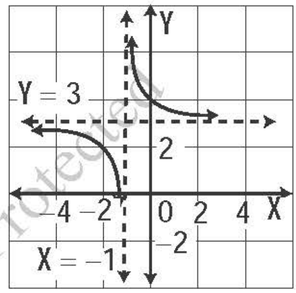
Let us draw the graph of $y = \dfrac{3}{x-1} + 2$.
This equation is defined for $x \neq 1$.
Hence vertical asymptote is $x = 1$.
Since $x = \dfrac{1+y}{y-2}$, horizontal asymptote is $y = 2$.
Domain $= \{x \mid x \neq 1,\ x \in \mathbb{R}\}$.
Range $= \{y \mid y \neq 2,\ y \in \mathbb{R}\}$.
If $x > 1$, then $y > 2$, since $\dfrac{3}{x-1} > 0$.
If $x < 1$, then $y < 2$, since $\dfrac{3}{x-1} < 0$.

(iii) The Graph of $y = \dfrac{ax+b}{cx+d}$

To draw the graph we need to transform the above function as that of case (ii).

Example 14.

Draw the graph of $y = \dfrac{3x+4}{x+1}$.



Solution

$$\frac{3x+4}{x+1} = \frac{3(x+1)+1}{x+1} = \frac{1}{x+1} + 3$$

We shall now draw the graph of $y = \dfrac{1}{x+1} + 3$.

As in case 2 we see that
vertical asymptote is $x = -1$,
horizontal asymptote is $y = 3$.
Domain $= \{x \mid x \neq -1,\ x \in \mathbb{R}\}$.
Range $= \{y \mid y \neq 3,\ y \in \mathbb{R}\}$.

Note: For the graph of $y = \dfrac{ax+b}{cx+d}$, the vertical asymptote is $x = -\dfrac{d}{c}$ and the horizontal asymptote is $y = \dfrac{a}{c}$.

Exercise 4.5

1. Sketch the graphs of the following functions.
   (a) $y = \dfrac{1}{x}$, (b) $y = \dfrac{3}{x}$, (c) $y = -\dfrac{2}{x}$, (d) $y = -\dfrac{1}{2x}$, (e) $y = \dfrac{1}{3x}$.
   State the domain and range of each function.

2. Sketch the graphs of:
   (a) $y = -\dfrac{2}{x} + 1$, (b) $y = \dfrac{2}{x-3}$, (c) $y = -\dfrac{1}{x+1} - 1$, (d) $y = \dfrac{2}{x+1} + 2$.
   State the domain and range of each function.

3. Sketch the graphs of:
   (a) $y = \dfrac{x+1}{x-1}$, (b) $y = \dfrac{-3x+4}{x-2}$, (c) $y = \dfrac{2x-3}{3x+1}$.
   State the domain and range of each function.

4.3.2 One-to-One Functions

Definition. A function f is called one-to-one

$$\text{if } x, y \in \text{dom}(f), \text{ and } x \neq y \text{ implies } f(x) \neq f(y).$$

In other words

$$\text{if } x, y \in \text{dom}(f), \text{ and } f(x) = f(y), \text{ then } x = y.$$

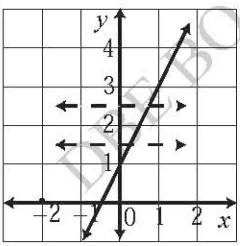
As an example, let $D = \{1, 2, 3\}$ and $R = \{3, 6, 9\}$. Define the function $f : D \to R$ such that $f(x) = 3x$. Then $f(1) = 3$, $f(2) = 6$, $f(3) = 9$. Thus we see that f is one-to-one.

It is easy to check whether the given function is one-to-one or not by using the horizontal line test.

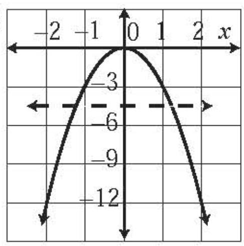
Horizontal Line Test: A real valued function is **one-to-one** if every horizontal line intersects the graph of the function at most one point.

Exercise 4.6

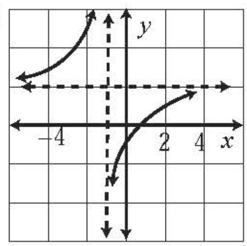
1. Determine whether each of the following function is a one-to-one function or not. If it is not one-to-one, explain why not.

(a)


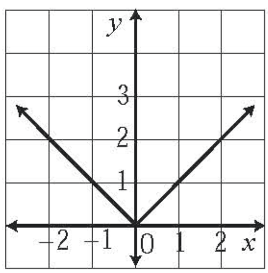
$f(x) = 2x + 1$

(b)


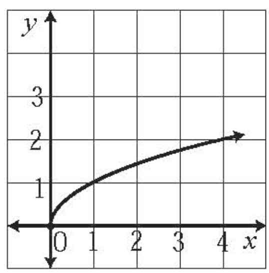
$f(x) = -3x^2$

(c)


$f(x) = \dfrac{2x-1}{x+1}$

(d)


$f(x) = |x|$

(e)


$f(x) = \sqrt{x}$

2. Draw the graph of the each given function and determine whether each is a one-to-one function or not.

(a) $f(x) = 3x + 2$ (b) $f(x) = x - 3$ (c) $f(x) = 4x^2$

(d) $f(x) = 2|x|$ (e) $f(x) = \dfrac{2x+3}{x+2}$ (f) $f(x) = 4x^2(0 \leq x \leq 4)$

(g) $f(x) = \sqrt{x}(x \geq 0)$.

4.3.3 Inverse Functions

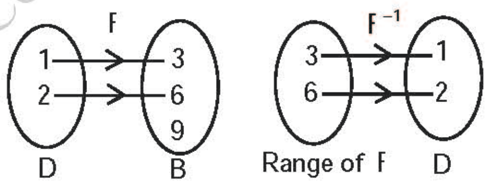
Definition. Let D be the domain and R be the range of function f given by $y = f(x)$. Suppose that f is a one-to-one function. The **inverse** function f^{-1} is defined by

$$f^{-1}(y) = x \quad \text{if and only if} \quad f(x) = y.$$

The domain of f^{-1} is R and the range of f^{-1} is D. The symbol f^{-1} is for the inverse of f and is read f **inverse**.

Example 15.

Let $D = \text{dom}(f) = \{1,2\}$. The function f is defined by $f(1) = 3$, $f(2) = 6$. Then its inverse $f^{-1} : \text{ran}(f) \to D$ is such that $f^{-1}(3) = 1$, $f^{-1}(6) = 2$.



Example 16.

Let the function f be given by $f(x) = 2x+3$ and let y be the image of x under f. Find the formula for f^{-1}.

Solution

Let $y = f(x)$. Then $y = 2x + 3$. This gives

$$2x = y - 3$$

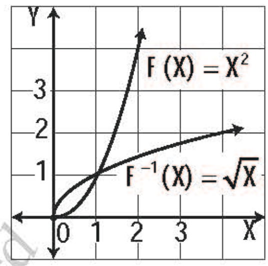
$$x = \frac{y-3}{2}.$$

$$\text{Hence,} \quad f^{-1}(y) = \frac{y-3}{2}.$$

Therefore $f^{-1}(x) = \dfrac{x-3}{2}$.

Example 17.

Let $f : [0, \infty) \to [0, \infty)$ be defined by $f(x) = x^2$ and let y be the image of x under f. Find the formula for f^{-1}.



Solution

Let $y = f(x)$. Thus $y = x^2$ gives $x = \pm\sqrt{y}$.
Since $\text{dom}(f) = \{x \mid x \geq 0\}$, $\text{ran}(f) = \{y \mid y \geq 0\}$, pick $f^{-1}(y) = \sqrt{y}$.
Therefore $f^{-1}(x) = \sqrt{x}$.

Example 18.

The function f is given by $f(x) = \dfrac{2}{3-4x}$. State the domain of f. Find the formula for f^{-1} and state the domain and range of f^{-1}.

Solution

$f(x) = \dfrac{2}{3-4x}$. Therefore $\text{dom}(f) = \{x \mid \ x \neq \dfrac{3}{4}, x \in \mathbb{R}\}$.

Let $f(x) = y$, then $\dfrac{2}{3-4x} = y$. This gives

$$4xy = 3y - 2$$
$$x = \frac{3y-2}{4y}$$
$$f^{-1}(y) = \frac{3y-2}{4y}.$$

Thus $f^{-1}(x) = \dfrac{3x-2}{4x}$.
Domain of $f^{-1} = \{x \mid x \neq 0, x \in \mathbb{R}\}$, Range of $f^{-1} = \{y \mid y \neq \frac{3}{4}, y \in \mathbb{R}\}$.

Example 19.

The function f is given by $f(x) = \dfrac{2x+3}{x-5}, x \neq 5$. Find $f^{-1}(3)$.

Solution

Let $a = f^{-1}(3)$. Then $f(a) = 3$.

$$\begin{aligned} \frac{2a+3}{a-5} &= 3 \\ 2a+3 &= 3a - 15 \\ a &= 18 \\ f^{-1}(3) &= 18 \end{aligned}$$

Exercise 4.7

1. Find the formula for f^{-1} and state the domain of f^{-1} when the function f is given by
(a) $f(x) = 2x - 3$, (b) $f(x) = 1 + 3x$, (c) $f(x) = 1 - x$,
(d) $f(x) = \dfrac{x+9}{2}$, (e) $f(x) = \frac{1}{3}(4x - 5)$, (f) $f(x) = \dfrac{2x+5}{x-7}$,
(g) $f(x) = \dfrac{3}{x-2}$, (h) $f(x) = \dfrac{13}{2x}$.

2. $A = \{x \mid x \geq 0,\ x \in \mathbb{R}\}$ and g, h are functions from A to A defined by $g(x) = 2x,\ h(x) = x^2$.
(a) Find the formula for the inverse functions $g^{-1},\ h^{-1}$.
(b) Evaluate $g^{-1}(7),\ h^{-1}(5)$.

3. Function f is given by $f(x) = \dfrac{2x-5}{x-3}$.
(a) State the value of x for which f is not defined.
(b) Find the value of x for which $f(x) = 0$.
(c) Find the inverse function f^{-1} and state the domain of f^{-1}.

4. Function f is given by $f(x) = \dfrac{x+a}{x-2}$ and that $f(7) = 2$, find
(a) the value of a, and (b) $f^{-1}(-4)$.

4.3.4 Composition of Functions

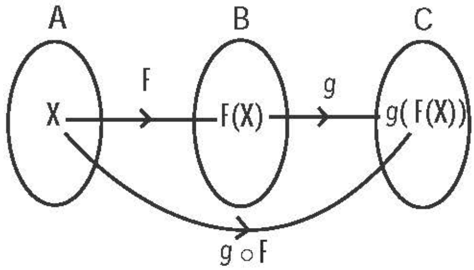
Definition. If f and g are functions such that $\text{ran}(f) \subset \text{dom}(g)$, then the **composite function** of f and g is the new function $g \circ f$ (read as g **circle** f) with $\text{dom}(g \circ f) = \text{dom}(f)$ such that

$$(g \circ f)(x) = g(f(x))$$

for all $x \in \text{dom}(f)$.

If $f : A \to B$ and $g : B \to C$ are given functions and x is any element in A, then we may illustrate the function $g \circ f$ as follows.

$$x \longmapsto f(x) \longmapsto g(f(x)).$$



Example 20.

Functions f and g are defined by $f(x) = x^2$ and $g(x) = 3x + 1$. Find
(a) $(g \circ f)(2)$ (b) $(f \circ g)(2)$ (c) $(g \circ f)(x)$ (d) $(f \circ g)(x)$.

Solution

(a) $(g \circ f)(2) = g(f(2)) = g(4) = 12 + 1 = 13.$

(b) $(f \circ g)(2) = f(g(2)) = f(7) = 7^2 = 49.$

(c) $(g \circ f)(x) = g(f(x)) = g(x^2) = 3x^2 + 1.$

(d) $(f \circ g)(x) = f(g(x)) = f(3x + 1) = (3x + 1)^2.$

Example 21.

Let the functions f and g be given by $f(x) = \dfrac{1}{2x-1}$ and $g(x) = \dfrac{1}{x+1}$.
(a) If dom $(f) = \{x \mid x \neq 0 \text{ and } x \neq \frac{1}{2}\}$, find $(g \circ f)(x)$.
(b) If dom $(g) = \{x \mid x \neq -1 \text{ and } x \neq 1\}$, find $(f \circ g)(x)$.
Solution

(a) $(g \circ f)(x) = g(f(x)) = g\left(\dfrac{1}{2x-1}\right) = \dfrac{1}{\frac{1}{2x-1} + 1} = \dfrac{2x-1}{2x}.$

$\therefore (g \circ f)(x) = \dfrac{2x-1}{2x}.$

(b) $(f \circ g)(x) = f(g(x)) = f\left(\dfrac{1}{x+1}\right) = \dfrac{1}{\frac{2}{x+1} - 1} = \dfrac{x+1}{1-x}.$

$\therefore (f \circ g)(x) = \dfrac{x+1}{1-x}.$

Example 22.

Let $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{x+1}{x-1}$.

(a) What is the domain of f for which the function $g \circ f$ exists? Find $(g \circ f)(x)$.

(b) What is the domain of g for which the function $f \circ g$ exists? Find $(f \circ g)(x)$.

Solution

(a) Since the function $g \circ f$ exists,

$$\begin{aligned}
\text{dom}(f) &= \text{dom}(g \circ f) \\
&= \{x \in \mathbb{R} \mid x \neq 0 \quad \text{and} \quad f(x) \neq 1\} \\
&= \{x \in \mathbb{R} \mid x \neq 0 \quad \text{and} \quad \frac{1}{x} \neq 1\} \\
&= \{x \in \mathbb{R} \mid x \neq 0 \quad \text{and} \quad x \neq 1\} \\
&= \mathbb{R} \setminus \{0, 1\}.
\end{aligned}$$

By definition, $(g \circ f)(x) = g(f(x)) = g\left(\dfrac{1}{x}\right) = \dfrac{\frac{1}{x}+1}{\frac{1}{x}-1}$.

$$\therefore (g \circ f)(x) = \frac{1+x}{1-x}.$$

(b) Since the function $f \circ g$ exists,

$$\begin{aligned}
\text{dom}(g) &= \text{dom}(f \circ g) \\
&= \{x \in \mathbb{R} \mid x \neq 1 \quad \text{and} \quad g(x) \neq 0\} \\
&= \{x \in \mathbb{R} \mid x \neq 1 \quad \text{and} \quad \frac{x+1}{x-1} \neq 0\} \\
&= \{x \in \mathbb{R} \mid x \neq 1 \quad \text{and} \quad x+1 \neq 0\} \\
&= \{x \in \mathbb{R} \mid x \neq 1 \quad \text{and} \quad x \neq -1\} \\
&= \mathbb{R} \setminus \{-1, 1\}.
\end{aligned}$$

By definition, $(f \circ g)(x) = f(g(x)) = f\left(\frac{x+1}{x-1}\right) = \dfrac{1}{\frac{x+1}{x-1}}$.

$$\therefore (f \circ g)(x) = \frac{x-1}{x+1}.$$

Properties of Composite Functions

1. The composition of linear functions is a linear function.

2. The composition of one-to-one functions is a one-to-one function.

3. Let $f : A \to B$, $g : B \to C$ and $h : C \to D$ be given functions. Then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

4. Let $f : A \to B$ be functions and $I_A : A \to A$ be an identity function on A. Then

$$f \circ I_A = f.$$

Moreover if $I_B : B \to B$ an identity function on B, then $I_B \circ f = f$.

Example 23.

Let f and g be defined by $f(x) = 2x + 3$ and $g(x) = 5x - 4$.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(5x - 4) \\ &= 2(5x - 4) + 3 \\ &= 10x - 5\end{aligned} \qquad \begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(2x + 3) \\ &= 5(2x + 3) - 4 \\ &= 10x + 11\end{aligned}$$

We see that both $f \circ g$ and $g \circ f$ are linear when f and g are linear functions.

Example 24.

Let the functions f, g and h be defined by $f(x) = 3x$, $g(x) = x - 1$ and $h(x) = x^2$.

$$\begin{aligned}(h \circ (g \circ f))(x) &= h((g \circ f)(x)) \\ &= h(g(f(x))) \\ &= h(g(3x)) \\ &= h(3x - 1) \\ &= (3x - 1)^2\end{aligned} \qquad \begin{aligned}((h \circ g) \circ f)(x) &= (h \circ g)(f(x)) \\ &= (h \circ g)(3x) \\ &= h(g(3x)) \\ &= h(3x - 1) \\ &= (3x - 1)^2\end{aligned}$$

Therefore $(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$ for all x.

Hence $h \circ (g \circ f) = (h \circ g) \circ f$.

Note. The composition of functions does not, in general, obey the commutative law.

For example, let us consider the functions f and g of Example 23.

We have $(f \circ g)(x) = 10x - 5.$
$(f \circ g)(1) = 5.$

But $(g \circ f)(x) = 10x + 11.$
$(g \circ f)(1) = 21.$

Hence $f \circ g \neq g \circ f$.

Exercise 4.8

1. Functions f and g are given by $f(x) = 2x + 1$ and $g(x) = 3x$.
   (a) Calculate $(g \circ f)(1)$ and $(g \circ f)(3)$.
   (b) Find the formula of $g \circ f$ and check the above images. State the domain of $g \circ f$.

2. The functions f and g are given by $f(x) = x + 2$ and $g(x) = x^2$.
   (a) Find the formulae for $g \circ f$, $g \circ g$, $f \circ g$, $f \circ f$ and their domains.
   (b) Find $(g \circ f)(-1)$ and $(g \circ f)(2)$.
   (c) Find $(f \circ g)(-1)$ and $(f \circ g)(2)$.

3. Find the formulae for composite functions $f \circ g$, $g \circ f$ and their domains in each case.
   (a) $f(x) = x + 1,$ $g(x) = 2x^2 - x + 3.$
   (b) $f(x) = x^2 - 1,$ $g(x) = 3x + 1.$
   (c) $f(x) = -x,$ $g(x) = x.$
   (d) $f(x) = x^2,$ $g(x) = \sqrt{x}.$

4. A function f is given by $f(x) = x + 1$. Find the function $g : \mathbb{R} \to \mathbb{R}$ in each of the following:
   (a) $(g \circ f)(x) = x^2 + 5x + 5$ (b) $(f \circ g)(x) = x^2 + 5x + 5.$

5. If $g : \mathbb{R} \to \mathbb{R}$ is given by $g(x) = x^2 + 3$, find the function f such that
   (a) $(f \circ g)(x) = 4x^2 + 3$ (b) $(g \circ f)(x) = 4x^2 + 3.$

6. Functions f and g are given by $f(x) = px - 2$ where p is a constant and $g(x) = 4x + 3$. Find the value of p for which $(f \circ g)(x) = (g \circ f)(x)$.

7. Let f and g be functions given by $f(x) = 3x - 1$ and $g(x) = x + 7$. Find the formulae of $f^{-1} \circ g$, $g^{-1} \circ f$ and state their domains. What are the values of $(f^{-1} \circ g)(3)$ and $(g \circ f^{-1})(2)$?

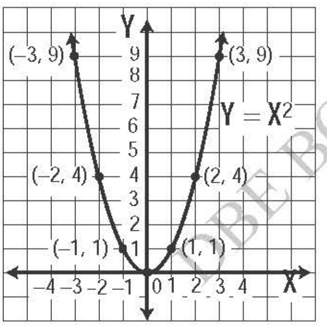
8. Let the functions f and g be given by $f(x) = 2x - 1$ and $g(x) = \dfrac{2x+3}{x-1}$.
   What is the domain of f for which the function $g \circ f$ exists? Find $(g \circ f)(x)$.
   Find the inverse function f^{-1} and g^{-1}.
   Evaluate $(f \circ g^{-1})(1)$ and $(f^{-1} \circ g^{-1})(1)$.

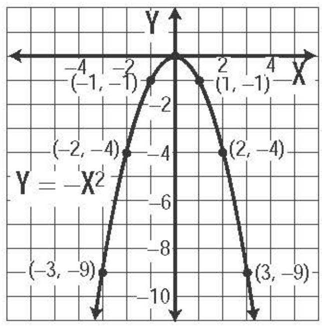
9. Let the functions f, g and h be $f(x) = x - 2$, $g(x) = x^3$ and $h(x) = 4x$. Show that $(h \circ g) \circ f = h \circ (g \circ f)$.

Chapter 5

Quadratic Functions

A function $y = ax^2 + bx + c$ where $a \neq 0$ is a **quadratic function** in the **standard form**. The graph of a quadratic function is called a **parabola**.





- If a is positive that is $a > 0$, then the parabola opens up as the graph of $y = x^2$.

- If a is negative that is $a < 0$, then the parabola opens down as the graph of $y = -x^2$.

Note that the graph of $y = -x^2$ is the **reflection on the x-axis** of the graph of $y = x^2$.

5.1 Graph of the Function $y = x^2 + bx + c$

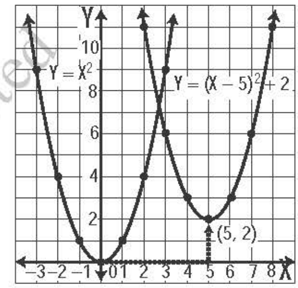
If $a = 1$ in the standard form, then the quadratic function is $y = x^2 + bx + c$. The graph of $y = x^2 + bx + c$ can be seen as the **translation** of the graph of $y = x^2$ as shown in the example below.

Example 1.

Consider the function $y = x^2 - 10x + 27$. Since

$$y = x^2 - 10x + 27 = (x - 5)^2 + 2;$$

the graph of $y = x^2 - 10x + 27$ is the translation of **positive** 5 units **horizontally** and **positive** 2 units **vertically** of the graph $y = x^2$.



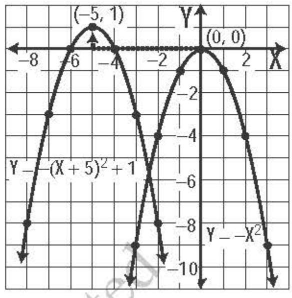
How to change $y = x^2 + bx + c$ to the form $y = (x - h)^2 + k$

$$\begin{aligned} y &= x^2 + bx + c \\ &= x^2 + bx + (\frac{b}{2})^2 - (\frac{b}{2})^2 + c \\ &= (x + \frac{b}{2})^2 - (\frac{b}{2})^2 + c \\ &= (x + \frac{b}{2})^2 - \frac{b^2 - 4c}{4} \\ &= (x - h)^2 + k \quad \text{where } h = -\frac{b}{2} \text{ and } k = -\frac{b^2 - 4c}{4}. \end{aligned}$$

5.2 Graph of the Function $y = -x^2 + bx + c$

If $a = -1$ in the standard form, then the quadratic function is $y = -x^2 + bx + c$. The graph of $y = -x^2 + bx + c$ can be seen as the **translation** of the graph of $y = -x^2$ as shown in the following example.

Example 2.



Consider the function $y = -x^2 - 10x - 24$. Since

$$y = -x^2 - 10x - 24 = -(x+5)^2 + 1$$

the graph of $y = -x^2 - 10x - 24$ is the translation of **negative** 5 units **horizontally** and **positive** 1 unit **vertically** of the graph $y = -x^2$.

How to change $y = -x^2 + bx + c$ to the form $y = -(x-h)^2 + k$

$$\begin{aligned} y &= -x^2 + bx + c \\ &= -(x^2 - bx) + c \\ &= -(x^2 - bx + (\frac{b}{2})^2 - (\frac{b}{2})^2) + c \\ &= -(x^2 - bx + (\frac{b}{2})^2) + (\frac{b}{2})^2 + c \\ &= -(x - \frac{b}{2})^2 + \frac{b^2 + 4c}{4} \\ &= -(x-h)^2 + k \quad \text{where } h = \frac{b}{2} \text{ and } k = \frac{b^2+4c}{4}. \end{aligned}$$

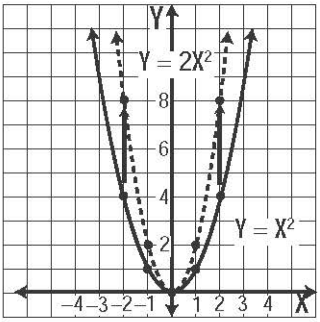
Exercise 5.1

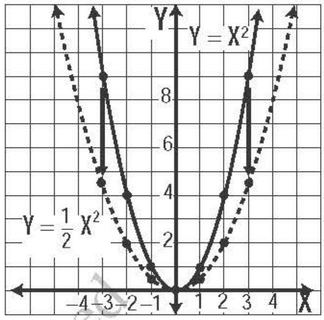
1. Compare the graphs of the following functions to the graph of $y = x^2$.
   (a) $y = x^2 + 2x + 3$ (b) $y = x^2 - 4x + 2$ (c) $y = x^2 + 4x + 4$

2. Compare the graphs of the following functions to the graph of $y = -x^2$.
   (a) $y = -x^2 + 2x + 3$ (b) $y = -x^2 - 4x - 7$ (c) $y = -x^2 + 4x - 4$

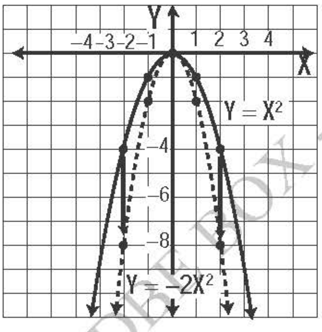
5.3 Graph of the Function $y = ax^2$

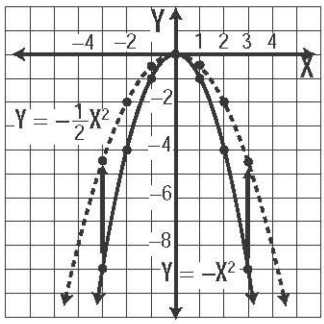
When a is **positive**, the graph of the function $y = ax^2$ is the vertical stretch of the scale factor a of the function $y = x^2$ as in the following figures.





When a is **negative**, the graph of the function $y = ax^2$ is the vertical stretch of the scale factor $|a|$ of the function $y = -x^2$ as in the following figures.





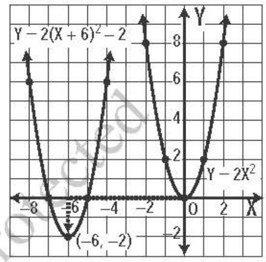
One can see that if the point (p, q) is on the graph $y = x^2$, then the point (p, aq) is on the graph $y = ax^2$. For example the point $(2, 4)$ is on the graph $y = x^2$, then

- the point $(2, 8)$ is on the graph $y = 2x^2$
- the point $(2, 2)$ is on the graph $y = \frac{1}{2}x^2$
- the point $(2, -8)$ is on the graph $y = -2x^2$
- the point $(2, -2)$ is on the graph $y = -\frac{1}{2}x^2$

5.4 Graph of the Function $y = ax^2 + bx + c$

The graph of $y = ax^2 + bx + c$ can be seen as the **translation** of the graph of $y = ax^2$ as shown in the examples below.

Example 3.

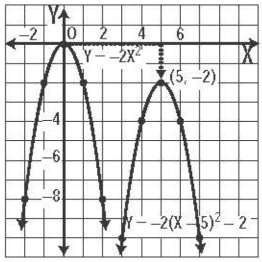


Consider the function $y = 2x^2 + 24x + 70$. Since

$$y = 2x^2 + 24x + 70 = 2(x + 6)^2 - 2$$

the graph of $y = 2x^2 + 24x + 70$ is the translation of **negative** 6 units **horizontally** and **negative** 2 units **vertically** of the graph $y = 2x^2$.

Example 4.



Consider the function $y = -2x^2 + 20x - 52$. Since

$$y = -2x^2 + 20x - 52 = -2(x - 5)^2 - 2$$

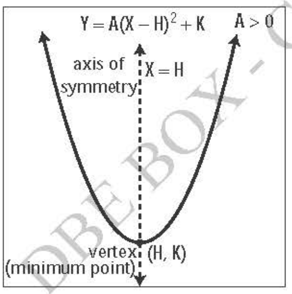
the graph of $y = -2x^2 + 20x - 52$ is the translation of **positive** 5 units **horizontally** and **negative** 2 units **vertically** of the graph $y = -2x^2$.

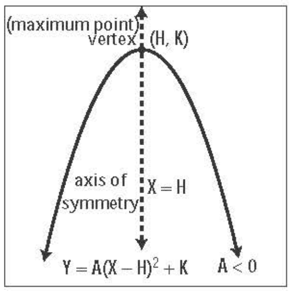
How to change $y = ax^2 + bx + c$ to the form $y = a(x - h)^2 + k$

$$\begin{aligned}
y &= ax^2 + bx + c \\
&= a(x^2 + \frac{b}{a}x) + c \\
&= a(x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 - (\frac{b}{2a})^2) + c \\
&= a(x^2 + \frac{b}{a}x + (\frac{b}{2a})^2) - a(\frac{b}{2a})^2 + c \\
&= a(x + \frac{b}{2a})^2 - \frac{b^2}{4a} + c \\
&= a(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a} \\
&= a(x - h)^2 + k \qquad \text{where } h = -\frac{b}{2a} \text{ and } k = -\frac{b^2 - 4ac}{4a}.
\end{aligned}$$

Features of the graph $y = ax^2 + bx + c = a(x-h)^2 + k$
• vertex: $(h, k) = (-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a})$
• axis of symmetry: $x = -\frac{b}{2a}$
• y-intercept: $(0, c)$
• domain = the set $\mathbb{R}$ of all real numbers
• range $= \begin{cases} \{y \mid y \geq k\} & \text{when } a > 0; k \text{ is minimum} \\ \{y \mid y \leq k\} & \text{when } a < 0; k \text{ is maximum.} \end{cases}$

The form $y = a(x-h)^2 + k$ is called the **vertex form** of the quadratic function.



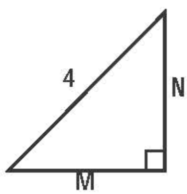


Example 5.

What is the largest area possible for a right triangle whose hypotenuse is 4 cm long?

Solution

Let m cm and n cm be two legs of the right triangle with hypotenuse 4 cm.



$$m^2 + n^2 = 4^2 = 16$$

$$n = \sqrt{16 - m^2}$$

Then area of the right triangle is $A = \frac{1}{2}mn$.

$$\begin{aligned} A &= \frac{1}{2}mn \\ &= \frac{1}{2}m\sqrt{16-m^2} \\ \therefore A^2 &= \frac{1}{4}m^2(16-m^2) \\ A^2 &= -\frac{1}{4}m^4 + 4m^2 \end{aligned}$$

Let $x = m^2$. Then

$$A^2 = -\frac{1}{4}x^2 + 4x.$$

The maximum value of A^2 occurs at $x = -\frac{4}{2(-\frac{1}{4})} = 8$.

So the maximum value of A occurs at $m = \sqrt{x} = \sqrt{8} = 2\sqrt{2}$.
When $m = 2\sqrt{2}$, $n = \sqrt{16-m^2} = 2\sqrt{2}$.
Since $\frac{1}{2}mn = \frac{1}{2}(2\sqrt{2})(2\sqrt{2}) = 4$, the largest area is 4 cm^2.

Exercise 5.2

1. Find the vertex form of each of the following quadratic functions. Find also y-intercept, axis of symmetry, vertex, and range of each of the functions.

   (a) $y = 2x^2 + 4x + 3$ (b) $y = 3x^2 - 6x + 2$ (c) $y = \frac{1}{2}x^2 + x - 4$
   (d) $y = -2x^2 + 2x + 3$ (e) $y = -3x^2 - 12x - 7$ (f) $y = -\frac{1}{2}x^2 - 3x - 4$

2. Find two positive numbers whose sum is 20 and whose product is as large as possible.

3. What is the largest area possible for a rectangle whose perimeter is 16 cm?

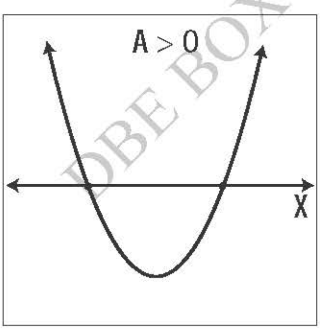
5.5 Discriminant of a Quadratic Function

The sign of the **discriminant** $b^2 - 4ac$ indicates whether or not the graph of the quadratic function passes through the x-axis.

First we consider the case $b^2 - 4ac > 0$.

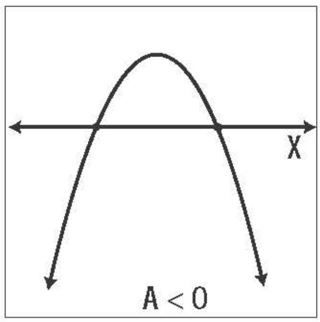
- If $a > 0$, then the parabola opens up and $-\dfrac{b^2 - 4ac}{4a} < 0$. Thus the vertex $(-\dfrac{b}{2a}, -\dfrac{b^2 - 4ac}{4a})$ is below the x-axis so the graph cuts the x-axis at two points.
- If $a < 0$, then the parabola opens down and $-\dfrac{b^2 - 4ac}{4a} > 0$. Thus the vertex $(-\dfrac{b}{2a}, -\dfrac{b^2 - 4ac}{4a})$ is above the x-axis so the graph cuts the x-axis at two points.

When $b^2 - 4ac > 0$ the graph passes through the x-axis at two points.



$Y = AX^2 + BX + C$

$B^2 - 4AC > 0$

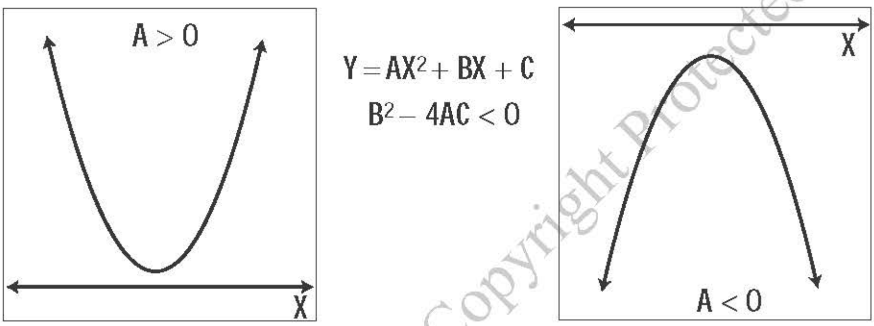


Next we consider the case $b^2 - 4ac < 0$.

- If $a > 0$, then the parabola opens up and $-\dfrac{b^2 - 4ac}{4a} > 0$. Thus the vertex $(-\dfrac{b}{2a}, -\dfrac{b^2 - 4ac}{4a})$ is above the x-axis so that the graph does not cut the x-axis.

- If $a < 0$, then the parabola opens down and $-\frac{b^2 - 4ac}{4a} < 0$. Thus the vertex $(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a})$ is below the x-axis so that the graph does not cut the x-axis.

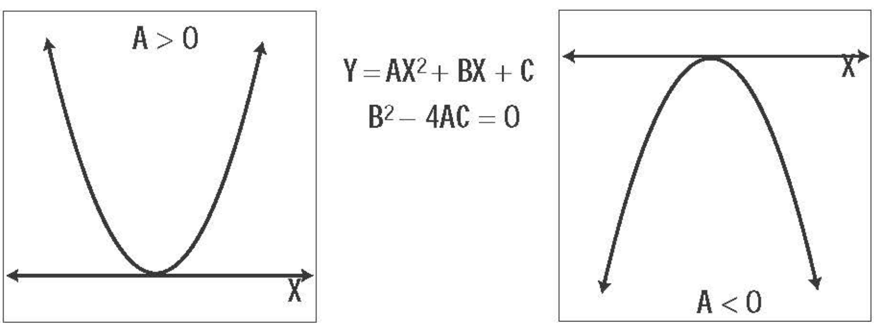
When $b^2 - 4ac < 0$ the graph does not pass through the x-axis.



Finally we consider the case $b^2 - 4ac = 0$.

When $b^2 - 4ac = 0$, the vertex $(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}) = (-\frac{b}{2a}, 0)$ is on the x-axis so the graph meets the x-axis at exactly one point.

When $b^2 - 4ac = 0$ the graph meets the x-axis at exactly one point.



If $b^2 - 4ac \geq 0$, the quadratic function $y = ax^2 + bx + c$ can be written as

$$y = a(x - p)(x - q).$$

This is called the **intercept form** or **factor form** of the quadratic function. In the intercept form, p and q are x-intercepts and the x-coordinate of the vertex of the quadratic function can be found as $\frac{p+q}{2}$.

Exercise 5.3

1. Find the discriminant of each of the following quadratic functions. Also find the number of x-intercepts of each of the functions.

   (a) $y = 3x^2 - 4x + 3$ (b) $y = 2x^2 - 4x - 3$ (c) $y = \frac{1}{2}x^2 + x - 4$
   (d) $y = -x^2 + 6x - 9$ (e) $y = -3x^2 - 12x - 7$ (f) $y = -\frac{1}{2}x^2 - 3x - 4$

2. Find the intercept form of each of the quadratic functions. Also find the y-intercept, axis of symmetry, vertex, and range of each of the functions.

   (a) $y = 2x^2 - 2x - 12$ (b) $y = 3x^2 - 6x + 3$ (c) $y = \frac{1}{2}x^2 + x - 4$
   (d) $y = 2x^2 - 5x - 3$ (e) $y = -6x^2 - 7x + 5$ (f) $y = -\frac{1}{2}x^2 - 3x - 4$

5.6 Quadratic Formula of $ax^2 + bx + c = 0$

Finding the x-intercepts of the quadratic function $y = ax^2 + bx + c$ is the same as finding the solutions of the quadratic equation $ax^2 + bx + c = 0$. We will derive the quadratic formula for the solutions of $ax^2 + bx + c = 0$ as in the following.

$$
\begin{aligned}
ax^2 + bx + c &= 0 \\
a(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a} &= 0 \\
a(x + \frac{b}{2a})^2 &= \frac{b^2 - 4ac}{4a} \\
(x + \frac{b}{2a})^2 &= \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}
$$

In this formula, we again find the discriminant $b^2 - 4ac$.

As stated in Section 5.5

- if $b^2 - 4ac > 0$, then the quadratic equation has two real solutions as
$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a};$$
- if $b^2 - 4ac = 0$, then the quadratic equation has only one real solution (repeated solution) as
$$x = -\frac{b}{2a};$$
- if $b^2 - 4ac < 0$, then the quadratic equation has no real solution.

Example 6.

Find the solutions for $2x^2 + 3x - 4 = 0$ by using quadratic formula.

Solution
Comparing the given equation $2x^2 + 3x - 4 = 0$ to the standard form $ax^2 + bx + c = 0$, we get $a = 2, b = 3, c = -4$. Then

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-3 \pm \sqrt{3^2 - 4(2)(-4)}}{2(2)} \\ &= \frac{-3 \pm \sqrt{9 + 32}}{4} \\ &= \frac{-3 \pm \sqrt{41}}{4} \\ x &= \frac{-3 + \sqrt{41}}{4} \quad \text{or} \quad x = \frac{-3 - \sqrt{41}}{4} \end{aligned}$$

Example 7.
Solve $2x^2 - 3x + 4 = 0$.
Solution
Comparing the given equation $2x^2 - 3x + 4 = 0$ to the standard form $ax^2 + bx + c = 0$, we get $a = 2, b = -3, c = 4$. Then

$$b^2 - 4ac = (-3)^2 - 4(2)(4) = 9 - 32 = -23 < 0$$

.
Therefore the equation has no real solution.

Exercise 5.4

Solve the following equations by using the quadratic formula.

1. $x^2 + 2x - 1 = 0$
2. $x^2 + 4x + 4 = 0$
3. $p^2 - 6p + 3 = 0$
4. $t^2 - 4t - 8 = 0$
5. $3q^2 - 12q + 11 = 0$
6. $5z^2 + 3z - 4 = 0$

5.7 Miscellaneous Exercises

There are problems that can be solved by creating quadratic equation by using the given information. In this section we will solve some examples.

Example 8.

Find the solution set of the system of equations:

$$\begin{aligned} x^2 + y^2 &= 25 \\ y &= x - 1 \end{aligned}$$

Solution

$$\begin{aligned} x^2 + y^2 &= 25 \qquad (1) \\ y &= x - 1 \qquad (2) \end{aligned}$$

Substituting $y = x - 1$ in (1), we get

$$\begin{aligned} x^2 + (x-1)^2 &= 25 \\ x^2 + x^2 - 2x + 1 &= 25 \\ 2x^2 - 2x - 24 &= 0 \\ x^2 - x - 12 &= 0 \\ (x+3)(x-4) &= 0 \\ x = -3 \quad &\text{or} \quad x = 4 \end{aligned}$$

Substituting $x = -3$ in (2), $y = -3 - 1 = -4$.

Substituting $x = 4$ in (2), $y = 4 - 1 = 3$.

Therefore the solution set is $\{(-3, -4), (4, 3)\}$.

Example 9.

Find the solution set of the system of equations:

$$\begin{aligned} x^2 - xy + 2y^2 &= 8 \\ 3x - 2y &= 2 \end{aligned}$$

Solution

$$\begin{aligned} x^2 - xy + 2y^2 &= 8 \qquad (1) \\ 3x - 2y &= 2 \qquad (2) \end{aligned}$$

Rewriting (2) in the equivalent form

$$y = \frac{3x-2}{2} \qquad (3)$$

Substituting $y = \dfrac{3x-2}{2}$ in (1),

$$\begin{aligned}
x^2 - x\frac{3x-2}{2} + 2\left(\frac{3x-2}{2}\right)^2 &= 8 \\
x^2 - \frac{x(3x-2)}{2} + \frac{2(3x-2)^2}{4} &= 8 \\
2x^2 - x(3x-2) + (3x-2)^2 &= 16 \\
2x^2 - 3x^2 + 2x + 9x^2 - 12x + 4 &= 16 \\
8x^2 - 10x - 12 &= 0 \\
4x^2 - 5x - 6 &= 0 \\
(4x+3)(x-2) &= 0 \\
x = -\frac{3}{4} \quad \text{or} \quad x = 2
\end{aligned}$$

Substituting $x = -\dfrac{3}{4}$ in (3),

$$y = \frac{3(-\frac{3}{4}) - 2}{2} = -\frac{17}{8}$$

Substituting $x = 2$ in (3),

$$y = \frac{3(2) - 2}{2} = 2$$

Hence the solution set is $\{(-\frac{3}{4}\ , -\frac{17}{8}), (2{,}2)\}$.

Example 10.

A rectangular garden has an area of 720 m^2. If the length is reduced by 6 m and the breadth is increased by 6 m, then the resulting area is the same as the original area. Find the length and breadth of the garden.

Solution

Suppose that the length of the garden is x m and the breadth of the garden is y m. Then the new length is $(x-6)$ m and the new breadth is $(y+6)$ m.

$$\begin{aligned} xy &= 720 \qquad (1)\\ (x-6)(y+6) &= 720 \qquad (2)\end{aligned}$$

Rewriting (2) in the equivalent form

$$\begin{aligned} xy+6x-6y-36 &= 720\\ xy+6x-6y &= 756 \qquad (3)\end{aligned}$$

Substituting $xy = 720$ in (3),

$$\begin{aligned} 720+6x-6y &= 756\\ 6x-6y &= 36\\ x-y &= 6\\ x &= 6+y \qquad (4)\end{aligned}$$

Substituting $x = 6+y$ in (1),

$$\begin{aligned} (6+y)y &= 720\\ y^2+6y &= 720\\ y^2+6y-720 &= 0\\ (y+30)(y-24) &= 0\\ y+30=0 \quad &\text{or} \quad y-24=0\\ y=-30 \quad &\text{or} \quad y=24\end{aligned}$$

Since y must be positive, $y=-30$ is impossible. So $y=24$.

Substituting $y=24$ in (4), $x = 24+6 = 30$.

Hence the length is 30 m and the breadth is 24 m.

Exercise 5.5

1. Find the solution set of each of the systems of equations:

   (a) $x^2 - y^2 = 9$
   $x + y = 1$

   (b) $y = \frac{8}{x}$
   $y = 7 + x$

   (c) $x^2 + 5x + y = 4$
   $x + y = 8$

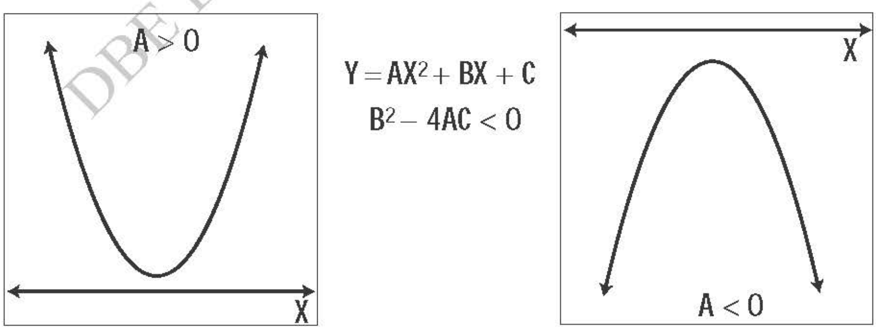
2. The sum of squares of two numbers is 58. If the first number and twice the second add up to 13, find the numbers.

3. The sum of the reciprocals of two positive numbers is $\frac{7}{36}$ and the product of the numbers is 108. Find the numbers.

5.8 Quadratic Inequality

From the discriminant $b^2 - 4ac$ of quadratic function $y = ax^2 + bx + c$ and sign of a, the coefficient of x^2, we can determine the solution set of x when the value of y is less than zero, equals zero, and is greater than zero.

Case 1. $\boxed{b^2 - 4ac < 0}$

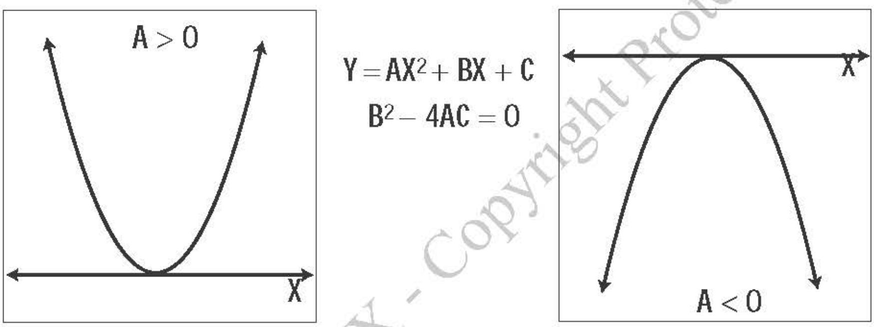


- If $b^2 - 4ac < 0$ and $a > 0$, the parabola opens up and does not cut the x-axis. Therefore all of the values of y are positive for all values of x.
- If $b^2 - 4ac < 0$ and $a < 0$, the parabola opens down and does not cut the x-axis. Therefore all of the values of y are negative for all values of x.

$a > 0$		$a < 0$	
	solution set		solution set
$y < 0$	$\varnothing$	$y < 0$	$\mathbb{R}$
$y = 0$	$\varnothing$	$y = 0$	$\varnothing$
$y > 0$	$\mathbb{R}$	$y > 0$	$\varnothing$

Case 2. $\boxed{b^2 - 4ac = 0}$

If $b^2 - 4ac = 0$, then $y = ax^2 + bx + c = a(x + \frac{b}{2a})^2$.



- If $b^2 - 4ac = 0$ and $a > 0$, the parabola opens up and meets the x-axis only at the vertex $(-\dfrac{b}{2a}, 0)$. Therefore all values of y are positive for all values of x except $-\dfrac{b}{2a}$. The value of y is zero, when $x = -\dfrac{b}{2a}$.
- If $b^2 - 4ac = 0$ and $a < 0$, the parabola opens down and meets the x-axis only at the vertex $(-\dfrac{b}{2a}, 0)$. Therefore all values of y are negative for all values of x except $-\dfrac{b}{2a}$. The value of y is zero, when $x = -\dfrac{b}{2a}$.

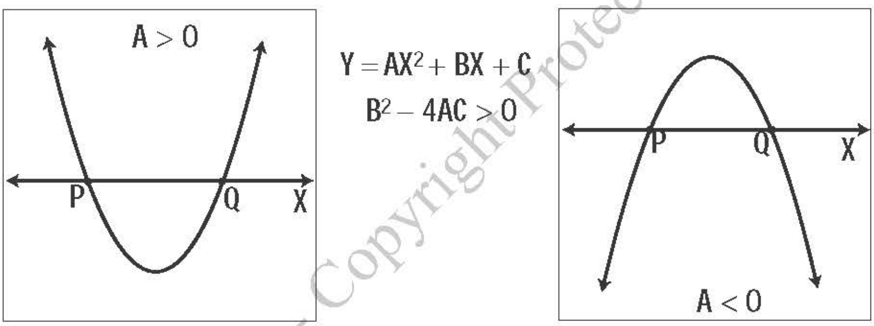
$a > 0$		$a < 0$	
	solution set		solution set
$y < 0$	$\varnothing$	$y < 0$	$\mathbb{R} \setminus \{-\frac{b}{2a}\}$
$y = 0$	$\{-\frac{b}{2a}\}$	$y = 0$	$\{-\frac{b}{2a}\}$
$y > 0$	$\mathbb{R} \setminus \{-\frac{b}{2a}\}$	$y > 0$	$\varnothing$

Case 3. $\boxed{b^2 - 4ac > 0}$

If $b^2 - 4ac > 0$, then

$$y = ax^2 + bx + c = a(x - p)(x - q),$$

where p and q are x-intercepts $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ of the function. Let us assume that $p < q$.



- If $a > 0$, the parabola opens up. The values of y are positive for $x < p$ or $x > q$. The values of y are equal to zero when $x = p$ or $x = q$. The values of y are negative for $p < x < q$.

- If $a < 0$, the parabola opens down. The values of y are positive for $p < x < q$. The values of y are equal to zero when $x = p$ or $x = q$. The values of y are negative for $x < p$ or $x < q$.

$a > 0$		$a < 0$	
	solution set		solution set
$y < 0$	$\{x \mid p < x < q\}$	$y < 0$	$\{x \mid x < p \text{ or } x > q\}$
$y = 0$	$\{p, q\}$	$y = 0$	$\{p, q\}$
$y > 0$	$\{x \mid x < p \text{ or } x > q\}$	$y > 0$	$\{x \mid p < x < q\}$

Example 11.

Find the solution set of each of the inequalities $2x^2+x+3<0$, $2x^2+x+3\leq 0$, $2x^2+x+3>0$, and $2x^2+x+3\geq 0$.

Solution
Let $y=2x^2+x+3$. Comparing the function $y=2x^2+x+3$ to the standard form $y=ax^2+bx+c$, we get $a=2, b=1, c=3$. Then

$$b^2-4ac=1-4(2)(3)=-23<0$$

Since $a>0$ and $b^2-4ac<0$, the parabola of the function opens up and does not cut the x-axis. Therefore all of the values of y are positive for all values of x.

The solution set of $2x^2+x+3<0$ is $\varnothing$.
The solution set of $2x^2+x+3\leq 0$ is $\varnothing$.
The solution set of $2x^2+x+3>0$ is $\mathbb{R}$.
The solution set of $2x^2+x+3\geq 0$ is $\mathbb{R}$.

Example 12.

Find the solution set of each of the inequalities $-2x^2+x-3<0$, $-2x^2+x-3\leq 0$, $-2x^2+x-3>0$, and $-2x^2+x-3\geq 0$.

Solution
Let $y=-2x^2+x-3$. Comparing the function $y=-2x^2+x-3$ to the standard form $y=ax^2+bx+c$, we get $a=-2, b=1, c=-3$. Then

$$b^2-4ac=1-4(-2)(-3)=-23<0$$

Since $a<0$ and $b^2-4ac<0$, the parabola of the function opens down and does not cut the x-axis. Therefore all of the values of y are negative for all values of x.

The solution set of $-2x^2+x-3<0$ is $\mathbb{R}$.
The solution set of $-2x^2+x-3\leq 0$ is $\mathbb{R}$.
The solution set of $-2x^2+x-3>0$ is $\varnothing$.
The solution set of $-2x^2+x-3\geq 0$ is $\varnothing$.

Example 13.

Find the solution set of each of the inequalities $2x^2+8x+8<0$, $2x^2+8x+8 \le 0$, $2x^2+8x+8>0$, and $2x^2+8x+8 \ge 0$.

Solution

Let $y = 2x^2+8x+8$. Comparing the function $y = 2x^2+8x+8$ to the standard form $y = ax^2+bx+c$, we get $a = 2, b = 8, c = 8$. Then

$$b^2 - 4ac = 64 - 4(2)(8) = 0;$$

$$y = 2x^2 + 8x + 8 = 2(x+2)^2$$

Since $a > 0$ and $b^2 - 4ac = 0$, the parabola of the function opens up and meets the x-axis only at the vertex $(-2, 0)$. Therefore all values of y are positive for all values of x except -2. The value of y is zero, when $x = -2$.

The solution set of $2x^2+8x+8<0$ is $\varnothing$.
The solution set of $2x^2+8x+8 \le 0$ is $\{-2\}$.
The solution set of $2x^2+8x+8>0$ is $\mathbb{R} \setminus \{-2\}$.
The solution set of $2x^2+8x+8 \ge 0$ is $\mathbb{R}$.

Example 14.

Find the solution set of each of the inequalities $-2x^2-6x+8<0$, $-2x^2-6x+8 \le 0$, $-2x^2-6x+8>0$, and $-2x^2-6x+8 \ge 0$.

Solution

Let $y = -2x^2 - 6x + 8$. Comparing the function $y = -2x^2 - 6x + 8$ to the standard form $y = ax^2+bx+c$, we get $a = -2, b = -6, c = 8$. Then

$$b^2 - 4ac = 36 - 4(-2)(8) = 100 > 0;$$

$$y = -2x^2 - 6x + 8 = -2(x+4)(x-1)$$

Since $a < 0$ and $b^2 - 4ac > 0$, the parabola of the function opens down and cuts the x-axis at the two points $(-4, 0)$ and $(1, 0)$. Therefore when $x < -4$ or $x > 1$, the values of y are negative; when $x = -4$ and $x = 1$, the values of y are equal to zero; when $-4 < x < 0$, the values of y are positive.

The solution set of $-2x^2-6x+8<0$ is $\{x|\ x < -4 \text{ or } x > 1\}$.
The solution set of $-2x^2-6x+8 \le 0$ is $\{x|\ x \le -4 \text{ or } x \ge 1\}$.
The solution set of $-2x^2-6x+8>0$ is $\{x|\ -4 < x < 1\}$.
The solution set of $-2x^2-6x+8 \ge 0$ is $\{x|\ -4 \le x \le 1\}$.

Exercise 5.6

Find the solution set of each of the quadratic inequality:

1. $2x^2 - 3x + 2 > 0$
2. $5x^2 + 2 < 0$
3. $12x^2 + 7x - 10 \leq 0$
4. $2x^2 + 7x - 4 > 0$
5. $2x^2 - 8x + 8 \geq 0$
6. $5x^2 + 3x - 2 > 0$

Chapter 6

Absolute Value Functions

For a real number x, the **absolute value** or **modulus of** x, which is written as $|x|$, is defined as follow:

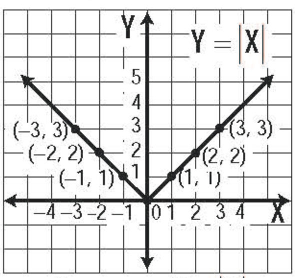
$$|x| = \begin{cases} x, & \text{when } x \geq 0, \\ -x, & \text{when } x < 0. \end{cases}$$

For example, $|3| = 3$ because $3 > 0$ and $|-3| = -(-3) = 3$ because $-3 < 0$.

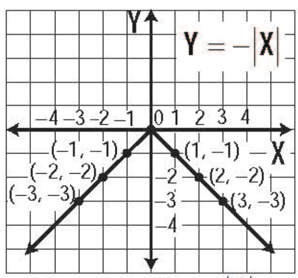
Absolute value of x can be seen as $|x| = \sqrt{x^2}$. For example,

$$|3| = \sqrt{3^2} = \sqrt{9} = 3 \text{ and } |-3| = \sqrt{(-3)^2} = \sqrt{9} = 3.$$

Functions like $y = |x|$ and $y = -|x|$ are called **absolute value functions**.



Graph of $Y = |X|$



Graph of $Y = -|X|$

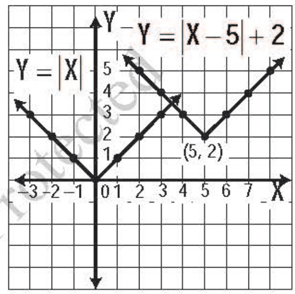
Note that the graph of $y = -|x|$ is the **reflection on the x-axis** of the graph of $y = |x|$.

6.1 Graph of the Function $y = |x - h| + k$

Graph of the absolute value function $y = |x - h| + k$ can be seen as the **translation** of h-units horizontally and k-units vertically of the graph $y = |x|$ as shown in the example below.

Example 1.

Consider the function $y = |x - 5| + 2$. The graph of $y = |x - 5| + 2$ is the translation of **positive** 5 units **horizontally** and **positive** 2 units **vertically** of the graph $y = |x|$.

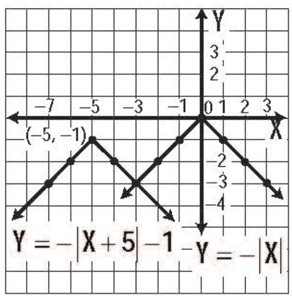


6.2 Graph of the Function $y = -|x - h| + k$

Graph of the absolute value function $y = -|x - h| + k$ can be seen as the **translation** of h-units horizontally and k-units vertically of the graph $y = -|x|$ as shown in the example below.

Example 2.

Consider the function $y = -|x + 5| - 1$. The graph of $y = -|x + 5| - 1$ is the translation of **negative** 5 units **horizontally** and **negative** 1 unit **vertically** of the graph $y = -|x|$.



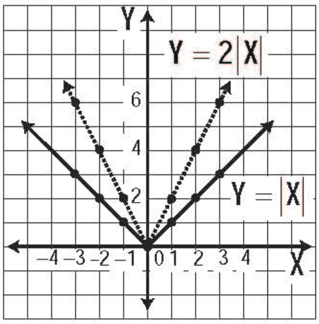
Exercise 6.1

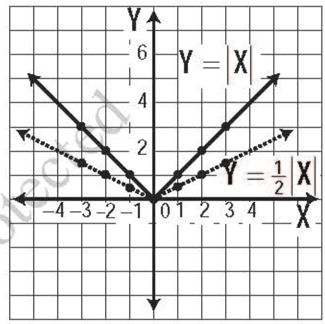
1. Compare the graphs of the following functions to the graph of $y = |x|$.
   (a) $y = |x - 3| - 2$ (b) $y = |x + 1| + 3$ (c) $y = |x - 2| + 3$

2. Compare the graphs of the following functions to the graph of $y = -|x|$.
   (a) $y = -|x + 3| + 2$ (b) $y = -|x - 4| + 1$ (c) $y = -|x + 4| - 1$

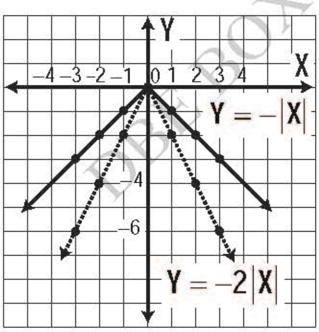
6.3 Graph of the Function $y = a|x|$

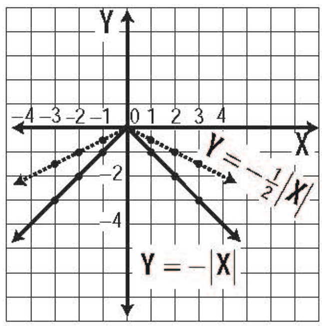
When a is **positive**, the graph of the function $y = a|x|$ is the vertical stretch of the scale factor a of the function $y = |x|$ as in the following figures.





When a is **negative**, the graph of the function $y = a|x|$ is the vertical stretch of the scale factor $|a|$ of the function $y = -|x|$ as in the following figures.





One can see that if the point (p, q) is on the graph $y = |x|$, then the point (p, aq) is on the graph $y = a|x|$. For example,

the point $(-2, 2)$ is on the graph $y = |x|$, then

- the point $(-2, 4)$ is on the graph $y = 2|x|$
- the point $(-2, 1)$ is on the graph $y = \frac{1}{2}|x|$

- the point $(-2, -4)$ is on the graph $y = -2|x|$
- the point $(-2, -1)$ is on the graph $y = -\frac{1}{2}|x|$

Since

$$\boxed{|pq| = \sqrt{(pq)^2} = \sqrt{p^2q^2} = \sqrt{p^2}\sqrt{q^2} = |p||q|},$$

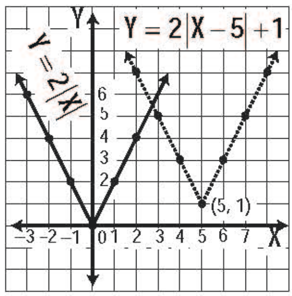
the function $y = |ax|$ can be seen as $y = |a||x|$.

6.4 Graph of the Function $y = a|x - h| + k$

The graph of the $y = a|x - h| + k$ can be seen as the **translation** of h-units horizontally and k-units vertically of the graph of $y = a|x|$ as shown in the examples below.

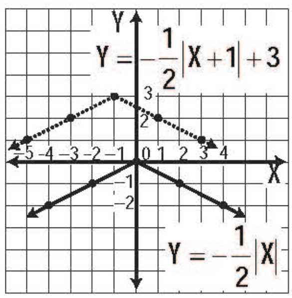
Example 3.

Consider the function $y = 2|x - 5| + 1$. The graph of $y = 2|x-5|+1$ is the translation of **positive** 5 units **horizontally** and **positive** 1 unit **vertically** of the graph $y = 2|x|$.



Example 4.

Consider the function $y = -\frac{1}{2}|x + 1| + 3$. The graph of $y = -\frac{1}{2}|x + 1| + 3$ is the translation of **negative** 1 unit **horizontally** and **positive** 3 units **vertically** of the graph $y = -\frac{1}{2}|x|$.



Features of the graph $y = a|x-h| + k$

- vertex: (h, k)
- axis of symmetry: $x = h$
- domain = the set $\mathbb{R}$ of all real numbers
- range $= \begin{cases} \{y \mid y \geq k\} & \text{when } a > 0 \\ \{y \mid y \leq k\} & \text{when } a < 0 \end{cases}$
- $\begin{cases} \text{when } a > 0, & k \text{ is the minimum value of } y \\ \text{when } a < 0, & k \text{ is the maximum value of } y \end{cases}$

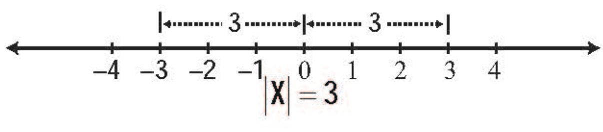
Exercise 6.2

Draw the graph of each of the following functions. Find also vertex, axis of symmetry, and range of each of the functions.

1. $y = |x| + 3$ 2. $y = -|x+3| - 2$ 3. $y = 3|x-4| + 1$
4. $y = -\frac{1}{2}|x| + 3$ 5. $y = \frac{1}{2}|x-4| + 1$ 6. $y = -\frac{1}{2}|x-3| - 1$

6.5 Equation $|x - p| = q$

By the definition of $|x|$, solutions of the equation $|x| = 3$ are $x = 3$ and $x = -3$. On the number line, the distance between 0 and x is $|x - 0| = |x|$. So solving equation like $|x| = 3$ is finding the points on number line which are distant 3 from 0.



Now we will consider the equation $|x - p| = q$.

- When $q < 0$, $|x - p| = q$ has no solution.
- When $q = 0$, $|x - p| = 0$ has only one solution p.

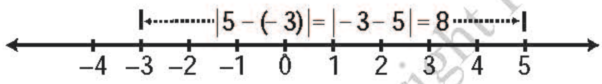
- When $q > 0$, the equation $|x - p| = q$ can be seen as

$$x - p = q \text{ or } x - p = -q.$$

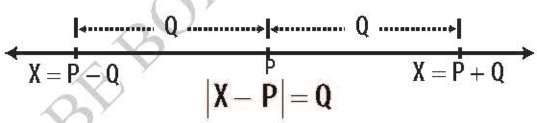
So the solutions are $x = p + q$ and $x = p - q$.

On the number line, the distance between p and q is $|p - q|$ or $|q - p|$.

For example, the distance between -3 and 5 can be calculated as $|-3-5| = 8$ or $|5-(-3)| = 8$.



On the number line, solving $|x - p| = q$ means finding the points x which are distant q from the point p.



Exercise 6.3

1. Find the solutions of the following equations. Illustrate each of the equations on the number line.

   (a) $|x - 5| = 3$ (b) $|x + 3| = 2$ (c) $|x - 4| = 1$

2. Find the solutions of the following equations.

   (a) $|2x - 5| = 4$ (b) $|-2x - 4| = 3$ (c) $|5x + 10| = 2$

6.6 Inequalities Involving $|x - p|$

In this section, inequalities of the forms

$$|x-p| < q, |x-p| \leq q, |x-p| > q \text{ and } |x-p| \geq q$$

will be considered. First note that

$$|x-p| = \begin{cases} x-p, & \text{when } x-p \geq 0, \\ -(x-p), & \text{when } x-p < 0. \end{cases}$$

Case 1. $|x-p| < q$

When $q \leq 0$, the inequality $|x-p| < q$ has no solution.

When $q > 0$, the inequality $|x-p| < q$ can be solved as follows:

- If $x-p \geq 0$, then $|x-p| = x-p < q$. So $0 \leq x-p < q$.
- If $x-p < 0$, then $|x-p| = -(x-p)$. So $-(x-p) < q$, thus $x-p > -q$. We have $-q < x-p < 0$.

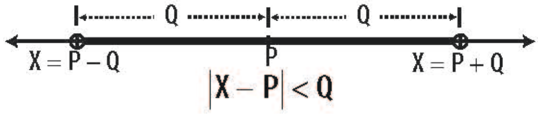
Therefore

$$-q < x-p < q$$

and then $p-q < x < p+q$.

The solution set of $|x-p| < q$ when $q > 0$ is $\{x | p-q < x < p+q\}$

Since $|x-p|$ is the distance between x and p, solving $|x-p| < q$ means finding the points x which are distant less than q from the point p.



Case 2. $|x-p| \le q$

When $q < 0$, the inequlaity $|x-p| \le q$ has no solution.

When $q \ge 0$, the solution set of $|x-p| \le q$ is $\{x|p-q \le x \le p+q\}$. It can be found as in the case of $|x-p| < q$.

The solution set of $|x-p| \le q$ when $q \ge 0$ is $\{x|\, p-q \le x \le p+q\}$.

Case 3. $|x-p| > q$

When $q < 0$, the solution set of $|x-p| > q$ is $\mathbb{R}$.

When $q \ge 0$, the inequality $|x-p| > q$ can be solved as follows:

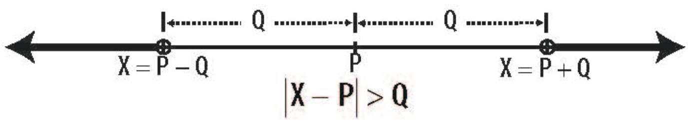
- If $x-p \ge 0$, then $|x-p| = x-p$. So $x-p > q$, and hence $x > p+q$.
- If $x-p < 0$, then $|x-p| = -(x-p)$. So $-(x-p) > q$, thus $x-p < -q$, and hence $x < p-q$.

Therefore

$$x < p-q \quad \text{or} \quad x > p+q.$$

The solution set of $|x-p| > q$, when $q \ge 0$ is $\{x|\, x < p-q \text{ or } x > p+q\}$.

Since $|x-p|$ is the distance between x and p, solving $|x-p| < q$ means finding the points x which are distant greater than q from the point p.



Case 4. $|x-p| \ge q$

When $q < 0$, the solution set of $|x-p| \ge q$ is $\mathbb{R}$.

When $q \ge 0$, the solution set of $|x-p| \ge q$ is $\{x|\, x \le p-q \text{ or } x \ge p+q\}$. It can be found as in the case of $|x-p| > q$.

The solution set of $|x-p| \geq q$, when $q \geq 0$ is $\{x| \, x \leq p-q \text{ or } x \geq p+q\}$.

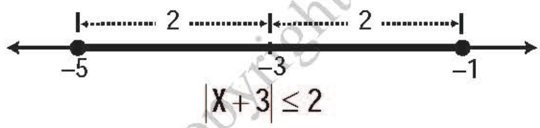
Example 5.

Find the solution set of the inequality $|x+3| \leq 2$. Illustrate the inequality on the number line.

Solution

$$|x+3| \leq 2$$
$$-2 \leq x+3 \leq 2$$
$$-5 \leq x \leq -1$$

The solution set is $\{x| -5 \leq x \leq -1\}$.



Example 6.

Find the solution set of the inequality $|2x-3| > 5$.

Solution

$$|2x-3| > 5$$
$$2x-3 < -5 \quad \text{or} \quad 2x-3 > 5$$
$$x < -1 \quad \text{or} \quad x > 4$$

The solution set is $\{x| \, x < -1 \text{ or } x > 4\}$.

Exercise 6.4

1. Find the solution sets of the following inequalities. Illustrate each of the inequalities on the number line.
   (a) $|x-1| < 3$ (b) $|x+5| > 2$ (c) $|x-2| \leq 4$
   (d) $|x+1| \geq 2$ (e) $|x-3| > 0$ (f) $|x+3| \leq 0$

2. Find the solution sets of the following inequalities.
   (a) $|2x-1| < 4$ (b) $|3x+5| > 6$ (c) $|4x-2| \leq 4$
   (d) $|2x+1| \geq 2$ (e) $|2x-3| > 0$ (f) $|5x+3| \leq 0$

Chapter 7

Probability

Probability as a general concept can be defined as the chance of an event occurring. Many people are familiar with probability from observation or playing games of chance, such as card games, slot machines or lotteries. In addition to being used in games of chance, probability theory is used in the fields of insurance, investments, weather forecasting and in various other areas. Blaise Pascal (1623-1662) and Pierre de Fermat (1601-1665) laid the foundation of probability theory.

7.1 Calculating Probability

In this chapter, we are concerned with experiments such as tossing a coin, rolling a die, drawing a ball from a bag and so on. These experiments are said to be *random experiments* or chance experiments (for brevity, we will simply call them *experiments* instead), since exact results of these experiments cannot be predicted. A single specific result of an experiment is called an *outcome*. The set of all possible outcomes of an experiment is called the *sample space*. An *event* is a subset of a sample space.

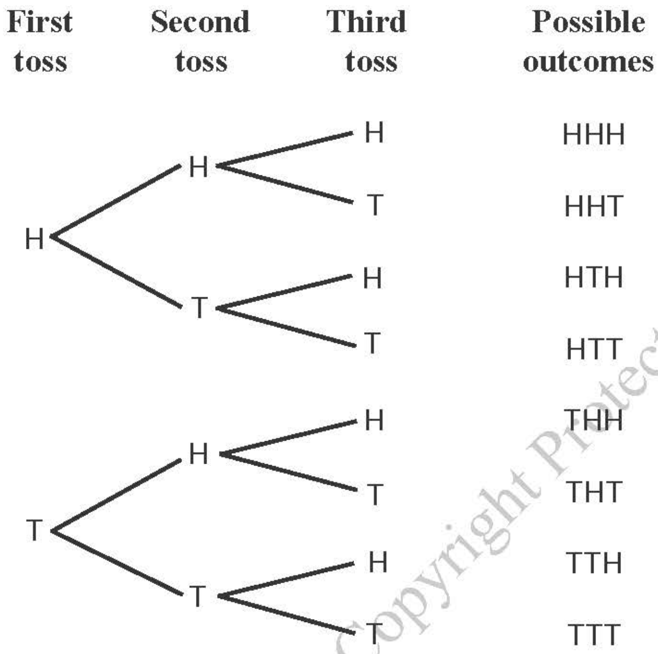
For the experiment "*tossing a coin*", there are two possible outcomes: head and tail (which will be denoted by H and T respectively). The sample space is {H, T}, and the events are ∅, {H}, {T} and {H, T}. But for the experiment "*tossing two coins*", the sample space is {(H, H), (H, T), (T, H), (T, T)} , which can also be expressed as {HH, HT, TH, TT}. The event {(H, T)} means that "*the first toss shows head and the second toss shows tail.*"

For the experiment "*rolling a die*", there are six possible outcomes: 1, 2, 3, 4, 5, 6. The sample space is {1, 2, 3, 4, 5, 6}. Some of the events are {3}, {4, 5, 6} and {2, 3, 5}, which can respectively be described as "*the result is* 3", "*the result is at least* 4" and "*the result is a prime number*".

The following **table** represents the sample space for rolling two dice. The first part of an ordered pair in the table represents the number appears on the first die and the second part represents the corresponding number on the second die. Sample spaces of similar experiments can also be obtained by constructing such tables.

		Second Die					
		1	**2**	**3**	**4**	**5**	**6**
First Die	**1**	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
	2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
	3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
	4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
	5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
	6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Sometimes, it is convenient to use a **tree diagram** to list all possible outcomes in a sample space. The following tree diagram displays the sample space for tossing a coin three times.



Let S be a finite sample space for an experiment such that all outcomes are *equally likely*, which means that they are random and have an equal likelihood of occurrence. Then the **probability of an event** A, denoted by $\mathrm{P}(A)$, in the sample space S, is defined by

$$\mathrm{P}(A) = \frac{\text{number of outcomes in the event } A}{\text{number of outcomes in the sample space } S}$$

In symbols,

$$\mathrm{P}(A) = \frac{\mathrm{n}(A)}{\mathrm{n}(S)}$$

where $\mathrm{n}(A)$ denotes the number of elements in the event A and $\mathrm{n}(S)$ denotes the number of outcomes in the sample space S.

Notice that:

- For any event A, P(A) is a real number such that $0 \leq P(A) \leq 1$.
- $P(\varnothing)=0$ and $P(S)=1$. (This means that the probability of an *impossible event* is 0 and that of a *sure event* is 1.)
- For any event A,
  $P(A) + P(\text{not}A) = 1$. **(Rule for complementary events)**

Example 1.

Find the probability of randomly selecting a red pen from a box that contains 2 red pens, 4 blue pens and 3 yellow pens.

Solution

Let A be the event of selecting a red pen.

$$P(A) = \frac{\text{number of outcomes in } A}{\text{number of outcomes in the sample space}} = \frac{2}{9}\ .$$

Example 2.

If a whole number from 1 to 20 both inclusive is randomly selected, and if each number has an equal chance of being selected, what is the probability that the number will be

(a) even? (b) greater than 1? (c) prime?

Solution

The sample space is $S = \{1, 2, 3, ..., 20\}$. $n(S) = 20$.

(a) The event is $E_1 = \{2, 4, 6, ..., 20\}$. $n(E_1) = 10$.

$$P(E_1) = \frac{n(E_1)}{n(S)} = \frac{10}{20} = \frac{1}{2}$$

(b) The event is $E_2 = \{2, 3, 4, ..., 20\}$. $n(E_2) = 19$.

$$P(E_2) = \frac{n(E_2)}{n(S)} = \frac{19}{20}$$

(c) The event is $E_3 = \{2, 3, 5, 7, 11, 13, 17, 19\}$. $n(E_3) = 8$.

$$P(E_3) = \frac{n(E_3)}{n(S)} = \frac{8}{20} = \frac{2}{5}$$

Example 3.

In a sample of 50 people, 21 had type O blood, 22 had type A blood, 5 had type B blood and, 2 had type AB blood. Find the following probabilities.

(a) A person has type O blood.

(b) A person has type A or type B blood.

(c) A person has neither type A nor type O blood.

(d) A person does not have type AB blood.

Solution

Type of blood	Frequency
A	22
B	5
AB	2
O	21
Total	50

(a) $P(O) = \frac{21}{50}$

(b) $P(A \text{ or } B) = \frac{27}{50}$

(c) $P(\text{neither A nor O}) = \frac{7}{50}$

(d) $P(\text{not AB}) = \frac{48}{50} = \frac{24}{25}$

Example 4.

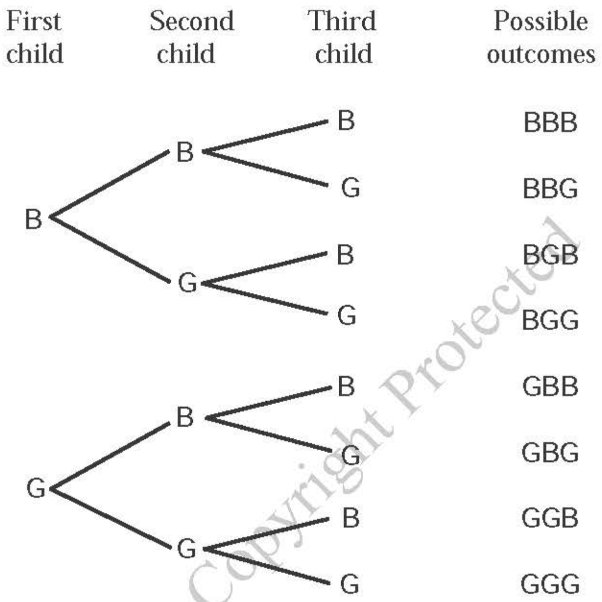
A family has three children. Find the probabilities of

(a) all boys, (b) exactly two boys,

(c) at most two boys, (d) at least one girl,

(e) at least one boy and at least one girl.

Solution



The sample space consists of 8 outcomes.

(a) The event is {BBB}.

P(all boys) $= \frac{1}{8}$

(b) The event is {BBG, BGB, GBB}.

P(exactly two boys) $= \frac{3}{8}$

(c) The event is { BBG, BGB, BGG, GBB, GBG, GGB, GGG}.

P(at most two boys) $= \frac{7}{8}$

(d) The event is {BBG, BGB, BGG, GBB, GBG, GGB, GGG}.

P(at least one girl) $= \frac{7}{8}$

(or) P(at least one girl) $= 1 - P(\text{all boys}) = 1 - \frac{1}{8} = \frac{7}{8}$

(by using rule for complementary events)

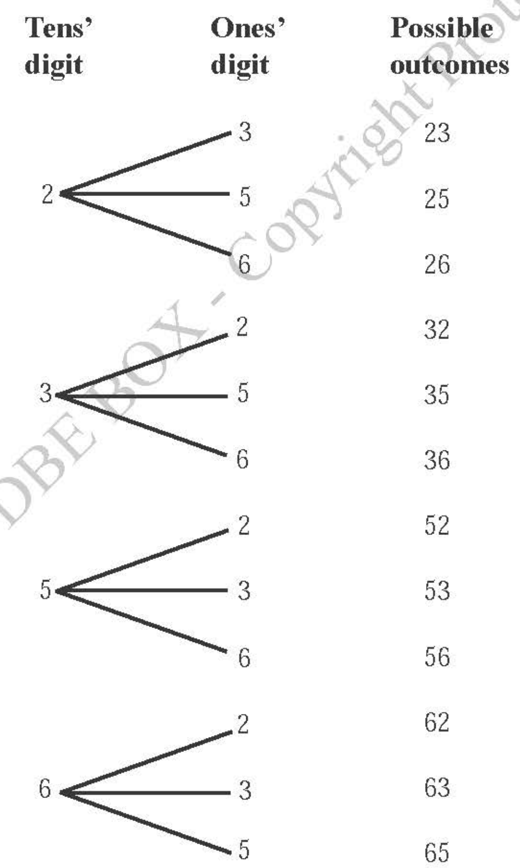
(e) The event is {BBG, BGB, BGG, GBB, GBG, GGB}.

P(at least one boy and at least one girl) $= \frac{6}{8} = \frac{3}{4}$

Example 5.

Draw a tree diagram to list all possible two-digit numerals which can be formed by using the digits 2, 3, 5 and 6 without repeating any digit. If one of these numerals is chosen at random, find the probability that it is divisible by 13. Find also the probability that it is either a prime or a perfect square. Find the probability that none of its digits is 6.

Solution



The sample space consists of 12 outcomes.

The event containing numerals divisible by 13 is $\{26, 52, 65\}$.

P(a numeral which is divisible by 13) $= \dfrac{3}{12} = \dfrac{1}{4}$

The event containing numerals which are primes or perfect squares is $\{23, 25, 36, 53\}$.

P(a numeral which is prime or a perfect square) $= \dfrac{4}{12} = \dfrac{1}{3}$

The event containing numerals which do not have digit 6 is $\{23, 25, 32, 35, 52, 53\}$.

P(a numeral none of its digits is 6) $= \dfrac{6}{12} = \dfrac{1}{2}$.

Example 6.

Two fair dice are thrown and the numbers appeared on top faces are recorded. Find the probability of each event:

(a) The first die shows 5.

(b) The sum of the numbers on the dice is 7.

(c) The product of the numbers on the two dice is greater than 24.

Solution

		Second Die					
		1	**2**	**3**	**4**	**5**	**6**
First Die	**1**	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
	2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
	3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
	4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
	5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
	6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

The sample space contains 36 outcomes.

(a) The event is $\{(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\}$.

P(5 on the first die) $= \dfrac{6}{36} = \dfrac{1}{6}$.

(b) The event is $\{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$.

P(the sum is 7) $= \dfrac{6}{36} = \dfrac{1}{6}$.

(c) The event is $\{(5, 5), (5, 6), (6, 5), (6, 6)\}$.
$$P(\text{product is greater than } 24) = \frac{4}{36} = \frac{1}{9}.$$

Exercise 7.1

1. A letter is chosen at random from the letters of the word ORANGE. What is the probability that it is a vowel?

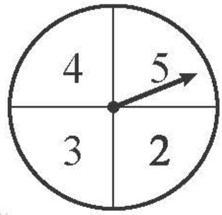
2. A bag contains 10 red balls and 30 black balls.
   (a) If a ball is drawn at random, what is the probability of getting a red ball?
   (b) Suppose the first ball drawn at random is red and is not replaced. If another ball is drawn at random, what is the probability that it will again be red?

3. How many three-digit numerals can be formed from 1, 5 and 7 without repeating any digit? Find the probability of a numeral which begins with 1.

4. A box contains five cards numbered as 2, 3, 4, 5 and 9. A card is chosen, the number is recorded, and the card is not replaced. Then another card is chosen and the number is recorded. Draw a tree diagram to get the possible outcomes. Find the probabilities of
   (a) getting two prime numbers,
   (b) getting two odd numbers and
   (c) getting a pair of numbers whose sum is a prime number.

5. A box contains four marbles of two blue, one red and one yellow. A marble is chosen, the colour is recorded, and the marble is not replaced. Then another marble is chosen and the colour is recorded. Draw a tree diagram to determine possible outcomes. Hence, find the probabilities of
   (a) getting two blue marbles and
   (b) getting two different colours.

6. A spinner is equally likely to point to any one of the numbers 2, 3, 4 and 5. Make a table of ordered pairs (first spin, second spin).
   Find the probability of
   (a) two odd numbers,
   (b) an even number followed by an odd number.



7. A coin is tossed and then a die is thrown. Head or tail and the number turns up are recorded each time. Draw a tree diagram and list the possible outcomes. Hence, find the probability that head and 6 turns up.

7.2 Probabilities of Combined Events

Two events are **independent** if the occurrence of any one of them does not affect the probability of the other. For example, when two fair coins are tossed, the event of getting head on the first coin and the event of getting tail on the second coin are independent.

Suppose an experiment consists of choosing a marble from a bag containing 3 red, 5 green, 2 blue and 6 yellow marbles, and rolling a die. Let us consider the events:

1. getting a blue marble from the bag

2. getting a prime number on the die

3. getting *both* a blue marble from the bag *and* a prime number on the die

Then the probability of the first event is $\frac{1}{8}$, the probability of the second event is $\frac{1}{2}$ and the probability of the third event is $\frac{1}{16}$. The first two events are independent. In fact, the product of the probabilities of the first two events *is the same as* the probability of the third (combined) event. The general rule for independent events is as follow:

Multiplication Rule :

A and B are *independent* events in a sample space *if and only if* $\text{P}(A \text{ and } B) = \text{P}(A) \times \text{P}(B)$.

Example 7.

A bag contains 3 red balls, 2 blue balls and 5 white balls. A ball is selected at random and its colour noted. Then it is replaced. A second ball is selected and its colour noted. Find the probabilities of:
(a) selecting 2 blue balls,
(b) selecting 1 blue ball and then 1 white ball,
(c) selecting 1 red ball and then 1 blue ball.

Solution

The bag contains 10 balls (3 red, 2 blue and 5 white).

(a)
$$\begin{aligned}\text{P(2 blue balls)} &= \text{P(1st ball is blue and 2nd ball is blue)}\\ &= \text{P(1st ball is blue)} \times \text{P(2nd ball is blue)}\\ &= \frac{2}{10} \times \frac{2}{10}\\ &= \frac{1}{25}\end{aligned}$$

(b)
$$\begin{aligned}\text{P(1 blue ball and then 1 white ball)} &= \text{P(1st ball is blue)} \times \text{P(2nd ball is white)}\\ &= \frac{2}{10} \times \frac{5}{10}\\ &= \frac{1}{10}\end{aligned}$$

(c)
$$\begin{aligned}\text{P(1 red ball then 1 blue ball)} &= \text{P(1st ball is red and 2nd ball is blue)}\\ &= \text{P(1st ball is red)} \times \text{P(2nd ball is blue)}\\ &= \frac{3}{10} \times \frac{2}{10}\\ &= \frac{3}{50}\end{aligned}$$

The events in above example are independent. But in some cases, the events considered *may not* be independent, that is, *the occurrence of the first event changes the probability of the occurrence of the second event.* These conditions can be seen in the following example.

Example 8.
A bag contains 9 red marbles and 3 green marbles. For each case below, find the probability of randomly selecting a red marble on the first draw and a green marble on the second draw.

(a) The first marble is replaced.
(b) The first marble is not replaced.

Solution

There are 12 marbles (9 red and 3 green).

(a) P(1st marble is red and 2nd marble is green)

$$= P(\text{1st marble is red}) \times P(\text{2nd marble is green})$$
$$= \frac{9}{12} \times \frac{3}{12} \quad = \frac{3}{16}$$

(b) P(1st marble is red and 2nd marble is green)

$$= P(\text{1st marble is red}) \times P(\text{2nd marble is green})$$
$$= \frac{9}{12} \times \frac{3}{11} \quad = \frac{9}{44}$$

Example 9.

At a teachers' conference, there are 4 English teachers, 3 Mathematics teachers, and 5 Science teachers. If 4 teachers are selected for a committee, find the probability that at least one is a science teacher.

Solution

There are 12 teachers (4 English, 3 Mathematics and 5 Science).

P(at least one is a Science teacher)

$$= 1 - P(\text{none of the 4 teachers is a science teacher})$$

=1− P(1st teacher is not a science teacher,
2nd teacher is not a science teacher,
3rd teacher is not a science teacher and
4th teacher is not a science teacher)

$$= 1 - \frac{7}{12} \times \frac{6}{11} \times \frac{5}{10} \times \frac{4}{9}$$

$$= 1 - \frac{7}{99} \quad = \frac{92}{99}$$

> **Addition Rule 1 :**
> For any two events A and B in a sample space,
> $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

Example 10.

In a hospital unit, there are 8 nurses and 5 doctors. Among them, there are 7 nurses and 3 doctors are females. If a staff person is selected, find the probability that the staff is a nurse or a male.

Solution

Staff	**Female**	**Male**	**Total**
Nurses	7	1	8
Doctors	3	2	5
Total	10	3	13

$$\begin{aligned} P(\text{nurse or male}) &= P(\text{nurse}) + P(\text{male}) - P(\text{male nurse}) \\ &= \frac{8}{13} + \frac{3}{13} - \frac{1}{13} \\ &= \frac{10}{13} \end{aligned}$$

Two events A and B are **mutually exclusive** if they cannot occur jointly, that is, they do not have common outcomes. For example, the occurrence of head and the occurrence of tail, in a single flip of a coin, are mutually exclusive.

If A and B are mutually exclusive events, then $P(A \text{ and } B) = 0$.

Addition Rule 2 :

If A and B are *mutually exclusive* events in a sample space, then $P(A \text{ or } B) = P(A) + P(B)$.

Example 11.

A box contains 3 strawberry doughnuts, 4 jelly doughnuts, and 5 chocolate doughnuts. If a doughnut is selected at random, find the probability that it is either a strawberry doughnut or a chocolate doughnut.

Solution

The box contains 12 doughnuts (3 strawberry, 4 jelly and 5 chocolate).

P(strawberry doughnut or chocolate doughnut)

$$= P(\text{strawberry doughnut}) + P(\text{chocolate doughnut})$$

$$= \frac{3}{12} + \frac{5}{12}$$

$$= \frac{8}{12} = \frac{2}{3}$$

Exercise 7.2

1. At a conference, there are 7 mathematics instructors, 5 computer science instructors, 3 statistics instructors and 4 science instructors. If an instructor is selected, find the probability of getting a science instructor or a math instructor.

2. Two dice are rolled. Find the probability of getting
   (a) a sum greater than 8 or a sum less than 3.
   (b) a product greater than 9 or a product less than 16.

3. A bag contains 15 discs of which 3 are white, 5 are red and 7 are blue. Two discs are to be drawn at random, in succession, each being replaced after its colour has been noted. Calculate the probability that the two discs will be of the same colour.

4. In a survey about a change in public policy, 100 people were asked if they favor the change, oppose the change, or have no opinion about the change. The responses are indicated as below:

	The Youth	Senior Citizens	Total
Favor	18	9	27
Oppose	12	25	37
No opinion	20	16	36
Total	50	50	100

Find the probability that a randomly selected respondent to this survey *oppose* or *has no opinion* about the change policy.

5. The probabilities that the student A and B pass an examination are $\frac{2}{3}$ and $\frac{3}{4}$ respectively. Find the probabilities that:
(a) both A and B pass the examination.
(b) exactly one of A and B passes the examination.

6. Three groups of children consist of 3 boys and 1 girl, 2 boys and 2 girls, and 1 boy and 3 girls respectively. If a child is chosen from each group, find the probability that 1 boy and 2 girls are chosen.

7.3 Calculation of Expected Frequency

The **expected frequency** of an event is the number of times that we predict the event will occur, in a given number of trials, based on a calculation using probabilities. It can be calculated by the formula:

$$\begin{matrix} Expected\ frequency \\ of\ an\ event \end{matrix} = \begin{matrix} Probability \\ of\ the\ event \end{matrix} \times \begin{matrix} Number \\ of\ trials \end{matrix}$$

Consider tossing a coin 1000 times. Since each toss is a trial, the number of trials is 1000. Since the probability of getting head in each trial is 0.5, the expected frequency of getting head is 500 $(= 0.5 \times 1000)$. This means that we can expect 500 heads occurring in 1000 trials. The expected frequency is based on probability. On the other hand, you can actually flip a coin 1000 times and observe the frequency of head. Do you think that these two frequencies are the same?

Example 12.

Traffic analysis found that the probability that a motorist will turn right at the intersection is $\frac{1}{3}$. Out of 300 motorists, how many would you expect to turn right at that intersection?

Solution

P(turning right) $= \frac{1}{3}$

Number of trials or motorists $= 300$

Expected number of motorists turning right at the intersection $= \frac{1}{3} \times 300$

$= 100$

Example 13.

A spinner is equally to point to any one of the numbers 1, 2, 3, 4, 5, 6, 7. What is the probability of scoring a number divisible by 3? If the arrow is spun 700 times, how many would you expect a number not divisible by 3?

Solution

Among the given numbers, the numbers 3 and 6 are divisible by 3.

P(a number divisible by 3) $= \frac{2}{7}$

P(a number not divisible by 3) $= 1 - \frac{2}{7} = \frac{5}{7}$

The number of trials $= 700$

Expected frequency of a number which is not divisible by 3 $= \frac{5}{7} \times 700$

$= 500.$

Exercise 7.3

1. After a large number of tossing a pin, the probability of 'pin up' was estimated to be 0.3. In 400 more trials, how many times would 'pin up' be expected?
2. If a die is rolled 60 times, what is the expected frequency of
   (a) 1 turns up?
   (b) a number divisible by 3 turns up?
   (c) a factor of 6 turns up?
3. Two honest coins are tossed. How many times would you expect to obtain two heads in 200 trials?
4. The probability of scoring 12 when throwing two dice at once is $\frac{1}{36}$. If such an experiment is repeated 720 times, what is the expected frequency of the score not being 12?
5. A spinner is equally likely to point to any one of the numbers: 1, 2, 3, ... , 10.
   (a) What is the probability of an odd number?
   (b) What is the probability of an even number?
   (c) If the arrow is spun 1000 times, what final score would you expect if all the individual scores are added together?

Chapter 8

Similarity

This chapter will be concerned with the study of the similarity of polygons especially triangles. We characterize the triangles which are similar by comparing their angles and their sides. We observe that the theorems on similar triangles depend very much on parallel lines that divides transverse line proportionally. Therefore ratio and proportion concepts should be reviewed before similarity.

The following properties of proportions are especially useful in geometry.

1. **The Means-extremes Product Property:** The product of the means equals the product of the extremes. If $\frac{a}{b} = \frac{c}{d}$ then $ad = bc$, where a, d are extremes and b, c are means. If $\frac{a}{b} = \frac{b}{c}$, then b is called the **geometric mean** of a and c.

2. **Invertendo Property:** In a proportion, the ratios may be inverted. If $\frac{a}{b} = \frac{c}{d}$ then $\frac{b}{a} = \frac{d}{c}$.

3. **Alternando Property:** In a proportion, the means (or extremes) may be interchanged. If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a}{c} = \frac{b}{d}$ (or $\frac{d}{b} = \frac{c}{a}$).

4. **Componendo Property:** If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a+b}{b} = \frac{c+d}{d}$.

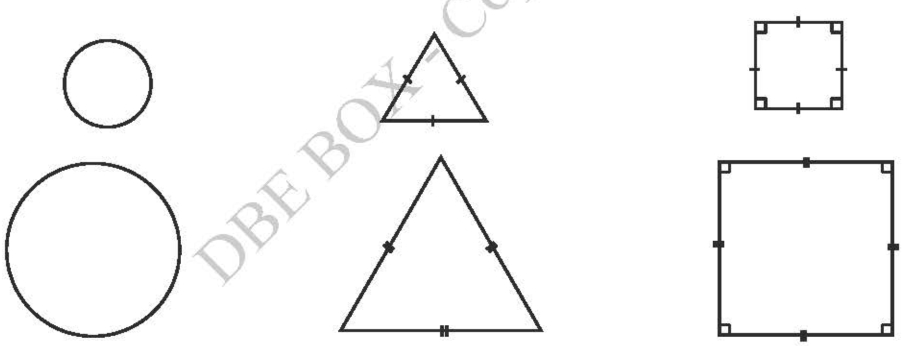
5. **Dividendo Property:** If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a-b}{b} = \frac{c-d}{d}$.

6. **Componendo and Dividendo Property:** If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

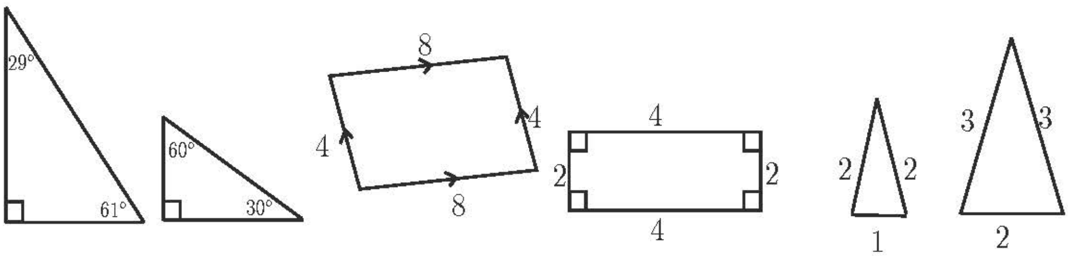
8.1 Ideas of Similarities and Similar Triangles

We now learn some properties of geometric figures that are of the same shape but not necessarily of the same size. Such figures are said to be similar.

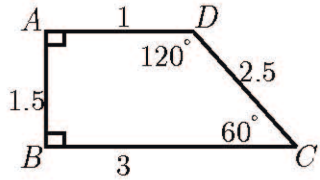
The following pairs of figures are similar.

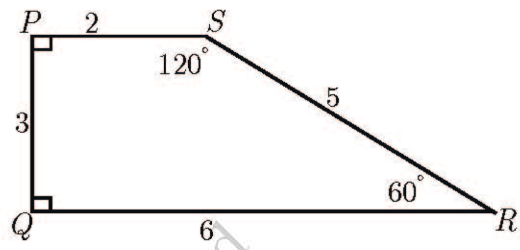


But the following pairs are not similar.



To consider the similarity of two figures we have to find the relation between their corresponding sides and corresponding angles.





Consider the correspondence $ABCD \leftrightarrow PQRS$, which matches A with P, B with Q, C with R and D with S.
We notice that the corresponding angles are equal, that is

$$\boxed{\angle A = \angle P, \angle B = \angle Q, \angle C = \angle R, \angle D = \angle S.}$$

Although corresponding sides are not equal by forming the ratios of their lengths, we have

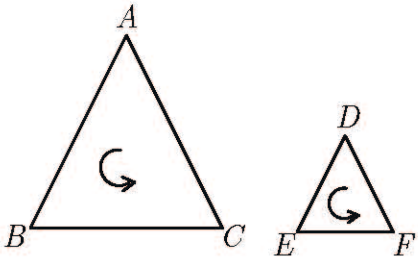
$$\frac{AB}{PQ} = \frac{1.5}{3} = \frac{1}{2}, \frac{BC}{QR} = \frac{3}{6} = \frac{1}{2}, \frac{CD}{RS} = \frac{2.5}{5} = \frac{1}{2}, \frac{AD}{PS} = \frac{1}{2}$$

$$\boxed{\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CD}{RS} = \frac{AD}{PS}.}$$

In general, if all the ratios of the lengths of the corresponding sides are equal, we say that the **corresponding sides are proportional.**

Thus the conditions for two polygons to be similar are
(1) corresponding angles must be equal, and
(2) corresponding sides must be proportional.

Definition. Two triangles whose corresponding angles are equal and whose corresponding sides are proportional are said to be **similar.**



That is, in $\triangle ABC$ and $\triangle DEF$,

if $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$, and $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF}$

then $\triangle ABC$ and $\triangle DEF$ are similar.

The symbol " $\sim$ " will be used to denote similarity.

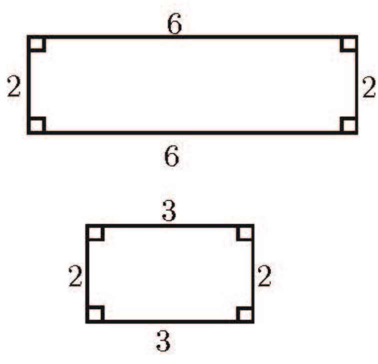
That is, if $\triangle ABC$ and $\triangle DEF$ are similar, we can write $\triangle ABC \sim \triangle DEF$.

As a convention, similar triangles and similar polygons will be named so that the order of letters indicates the correspondence between the two figures. **Thus the statement $\triangle ABC \sim \triangle DEF$ will always indicate that which angles are corresponding angles and which sides are corresponding sides.**

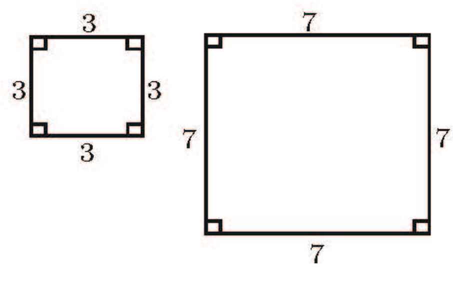
Exercise 8.1

1. State why the two polygons are, or are not, similar.

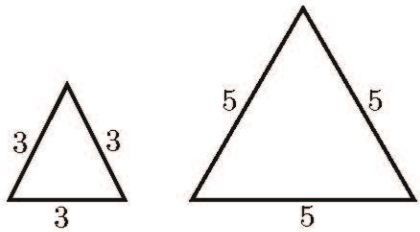
(a)



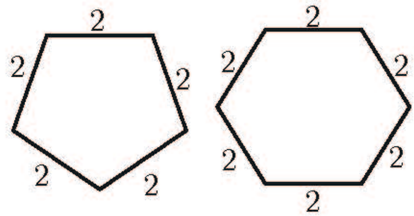
(b)



(c)



(d)



2. Complete the proportions.

   (a) If $\triangle ABC \sim \triangle DEF$ then $\dfrac{AB}{?} = \dfrac{BC}{?} = \dfrac{?}{DF}$.

   (b) If $\triangle GHI \sim \triangle KLM$ then $\dfrac{?}{HI} = \dfrac{?}{GH} = \dfrac{?}{GI}$.

3. State whether the proportions are correct for the indicated similar triangles.

   (a) $\triangle ABC \sim \triangle XYZ$

   $$\frac{AB}{XY} = \frac{BC}{YZ}$$

   (b) $\triangle DEF \sim \triangle HIJ$

   $$\frac{DE}{HI} = \frac{EF}{IJ}$$

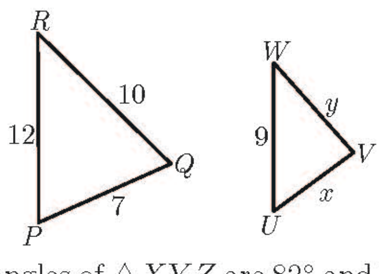
   (c) $\triangle RST \sim \triangle LMK$

   $$\frac{RT}{LM} = \frac{ST}{MK}$$

   (d) $\triangle XYZ \sim \triangle UVW$

   $$\frac{XY}{UV} = \frac{XZ}{VW}$$

4. Given : $\triangle PQR \sim \triangle UVW$ and lengths of sides are as marked.
   Find : The values of x and y.

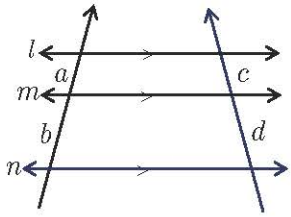


5. The measures of two angles of $\triangle XYZ$ are 82° and 16°. Find the measures of the angles of a triangle similar to $\triangle XYZ$.

8.2 The Basic Proportionality Theorem

Postulate 1. If three parallel lines intersect two transversals, then the lines divide the transversals proportionally.

According to the above postulate, if $l \parallel m \parallel n$, then $\frac{a}{b} = \frac{c}{d}$.

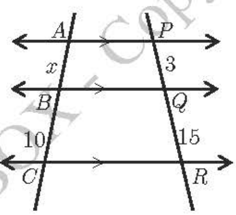


Example 1.

In the following figure if $AP \parallel BQ \parallel CR$, find the value of x.

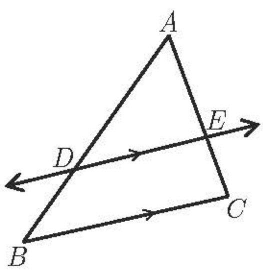
Solution

$$\frac{x}{10} = \frac{3}{15}$$
$$\therefore x = 2.$$



Theorem 1 (The Basic Proportionality Theorem-BPT). If a line intersecting the interior of a triangle is parallel to one side, then the line divides the other two sides proportionally.

$$\boxed{\text{In}\triangle ABC, \text{if } DE \parallel BC \text{ then } \frac{AD}{DB} = \frac{AE}{EC}.}$$



Corollary 1.1. Using properties of proportions, it can be shown that the following three proportions are equivalent that is they have same value.

(1) $\frac{AD}{DB} = \frac{AE}{EC}$

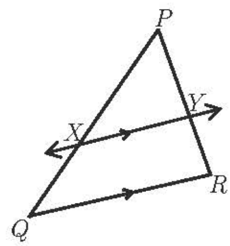
(2) $\frac{AD}{AB} = \frac{AE}{AC}$

(3) $\frac{AB}{DB} = \frac{AC}{EC}$

The following corollary is the converse of the BPT.

Corollary 1.2 (CBPT). If a line divides two sides of a triangle proportionally, then the line is parallel to the third side.

In$\triangle PQR$, if $\frac{PX}{XQ} = \frac{PY}{YR}$ then $XY \parallel QR$.



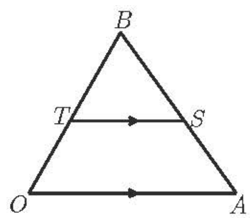
Exercise 8.2

1. Use the Basic Proportionality Theorem (BPT) and its corollary to complete the proportions for the adjoining figures.

In $\triangle AOB, TS \parallel OA$.

(a) $\frac{OT}{TB}$ =?

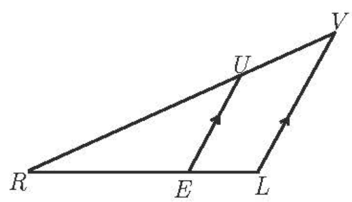
(b) $\frac{SA}{BA}$ =?



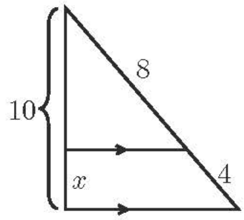
In $\triangle RVL, EU \parallel LV$.

(c) $\frac{EL}{RE}$ =?

(d) $\frac{RU}{RV}$ =?



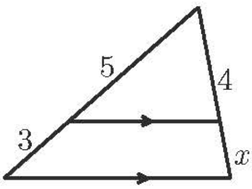
2. To find the value of x in the figure, a student wrote the proportion $\frac{x}{10} = \frac{4}{8}$.
   (a) Is this correct?
   (b) Another student wrote the proportion $\frac{x}{x-4} = \frac{4}{8}$. Is this correct?



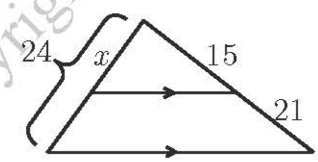
   (c) Write a simpler proportion that will give the correct answer.

3. Find the value of x in each of the figure below. (They are not drawn to scales.)

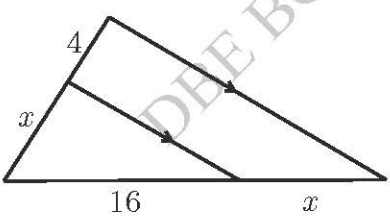
(a)



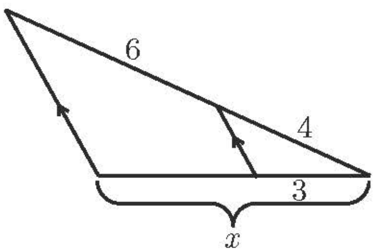
(b)



(c)



(d)

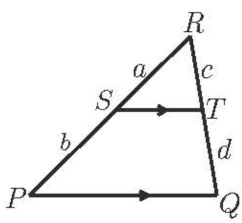


4. Given $\triangle PQR$ with $ST \parallel PQ$and lengths of segments are marked. Which of the following proportions are correct?

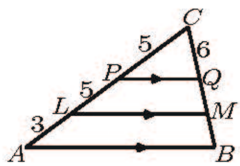
(a) $\frac{b}{a} = \frac{d}{c}$ (b) $\frac{a+b}{a} = \frac{c+d}{d}$

(c) $\frac{c}{d+c} = \frac{a}{b+a}$ (d) $\frac{a}{c} = \frac{b}{d}$

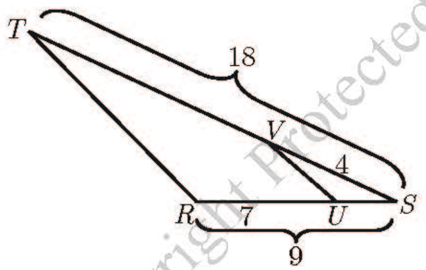
(e) $\frac{a}{b} = \frac{c}{d}$ (f) $\frac{a-b}{b} = \frac{c-d}{c}$.



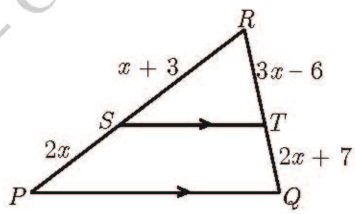
5. If $PQ \parallel LM \parallel AB$, and the lengths are as shown, how long are the segments MQ and BM?



6. If the segments in the figure have the lengths indicated, is $UV \parallel RT$? Justify your answer.

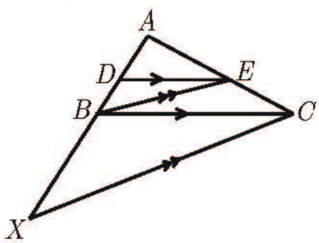


7. Given the figure as marked with $ST \parallel PQ$, find the lengths of the segments PS, SR, RT and TQ.



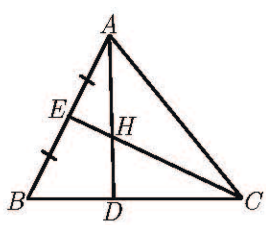
8. Given : $\dfrac{AD}{DB} = \dfrac{2}{1}$

   Prove : $\dfrac{AX}{XB} = \dfrac{3}{1}$



9. Given : $AE = EB$, $\dfrac{BD}{DC} = \dfrac{2}{3}$

   Find : The ratio $\dfrac{CH}{CE}$

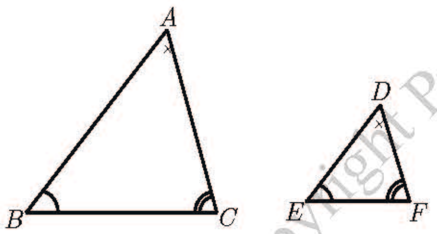


8.3 Basic Theorems on Similar Triangles

In this section we will study some theorems and properties of similar triangles.

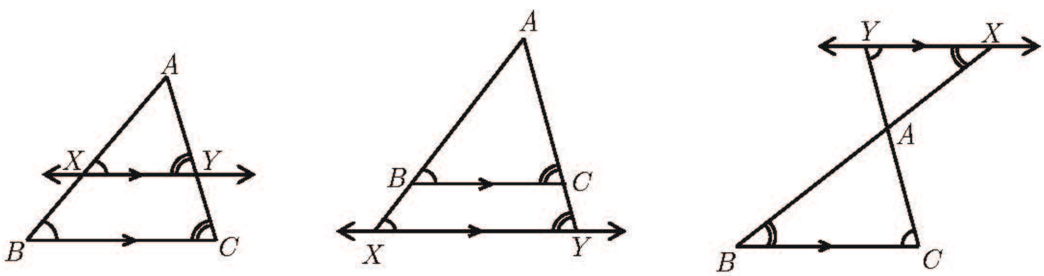
Theorem 2 (The AAA Similarity Theorem). If the angles of a triangle are equal to the angles of another triangle, then the two triangles are similar.

In $\triangle ABC$ and $\triangle DEF$, if $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$ then $\triangle ABC \sim \triangle DEF$.



Corollary 2.1 (The AA Corollary). If two angles of a triangle are equal to two angles of another triangle, then the triangles are similar.

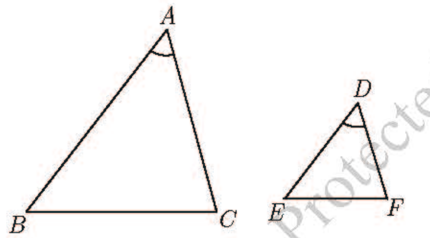
Corollary 2.2. If a line parallel to one side of a triangle determines a second triangle, then the second triangle will be similar to the original triangle.



In the above figures, lines XY are drawn parallel to BC forming another triangle AXY. We see that $\triangle ABC \sim \triangle AXY$, since indicated angles are equal.

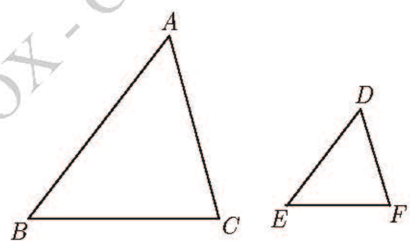
Theorem 3 (The SAS Similarity Theorem). If an angle of a triangle is equal to an angle of another triangle, and the sides including these angles are proportional, then the triangles are similar.

In $\triangle ABC$ and $\triangle DEF$, if $\angle A = \angle D$ and $\dfrac{AB}{DE} = \dfrac{AC}{DF}$ then $\triangle ABC \sim \triangle DEF$.



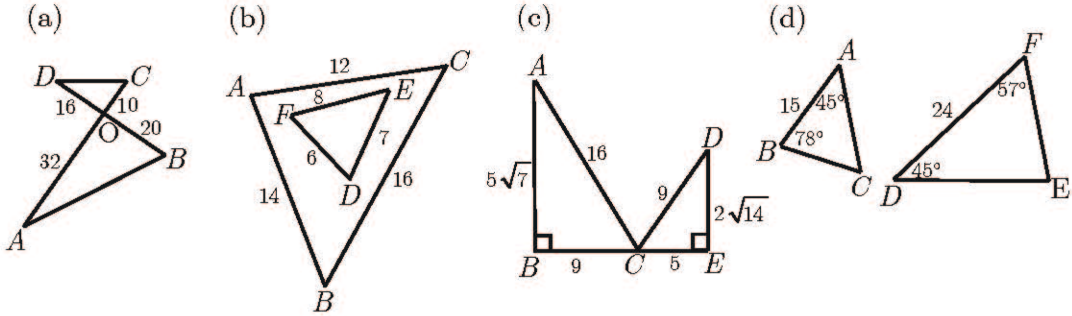
Theorem 4 (The SSS Similarity Theorem). If the corresponding sides of two triangles are proportional, then the triangles are similar.

In $\triangle ABC$ and $\triangle DEF$, if $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{CA}{FD}$ then $\triangle ABC \sim \triangle DEF$.



Example 2.

Determine whether the following pairs of triangle are similar or not. If they are similar, state why.



Solution

(a) Similar because $\angle AOB = \angle DOC$,

$$\frac{AO}{DO} = \frac{32}{16} = \frac{2}{1}, \qquad \frac{OB}{OC} = \frac{20}{10} = \frac{2}{1}$$

$$\therefore \frac{AO}{DO} = \frac{OB}{OC}$$

$$\therefore \triangle AOB \sim \triangle DOC(SAS).$$

(b) Similar because

$$\frac{DE}{AB} = \frac{7}{14} = \frac{1}{2}, \quad \frac{EF}{BC} = \frac{8}{16} = \frac{1}{2} \quad \frac{DF}{AC} = \frac{6}{12} = \frac{1}{2}.$$

$$\therefore \frac{DE}{AB} = \frac{EF}{BC} = \frac{DF}{AC}.$$

$$\therefore \triangle DEF \sim \triangle ABC(SSS).$$

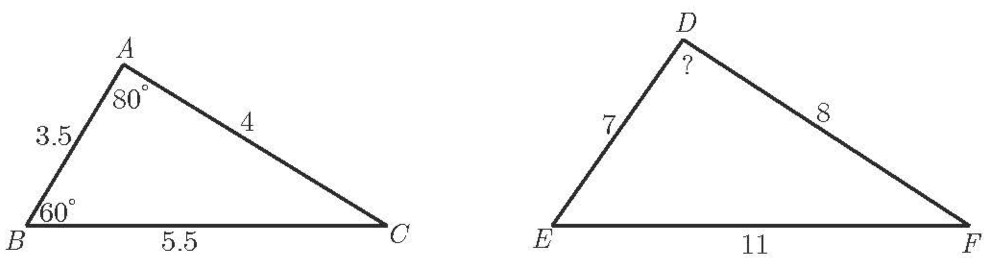
(c) Not similar.

(d) Similar because

$$\angle C = 180° - (45° + 78°) = 57°$$
$$\angle E = 180° - (45° + 57°) = 78°$$
$$\angle A = \angle D, \angle B = \angle E, \angle C = \angle F$$
$$\therefore \triangle DEF \sim \triangle ABC(AAA).$$

Example 3.

Find the value of angle D.



Solution. In $\triangle ABC$ and $\triangle DEF$,

$$\frac{AC}{DF} = \frac{4}{8} = \frac{1}{2},\ \frac{AB}{DE} = \frac{3.5}{7} = \frac{1}{2},\ \frac{BC}{EF} = \frac{5.5}{11} = \frac{1}{2}$$

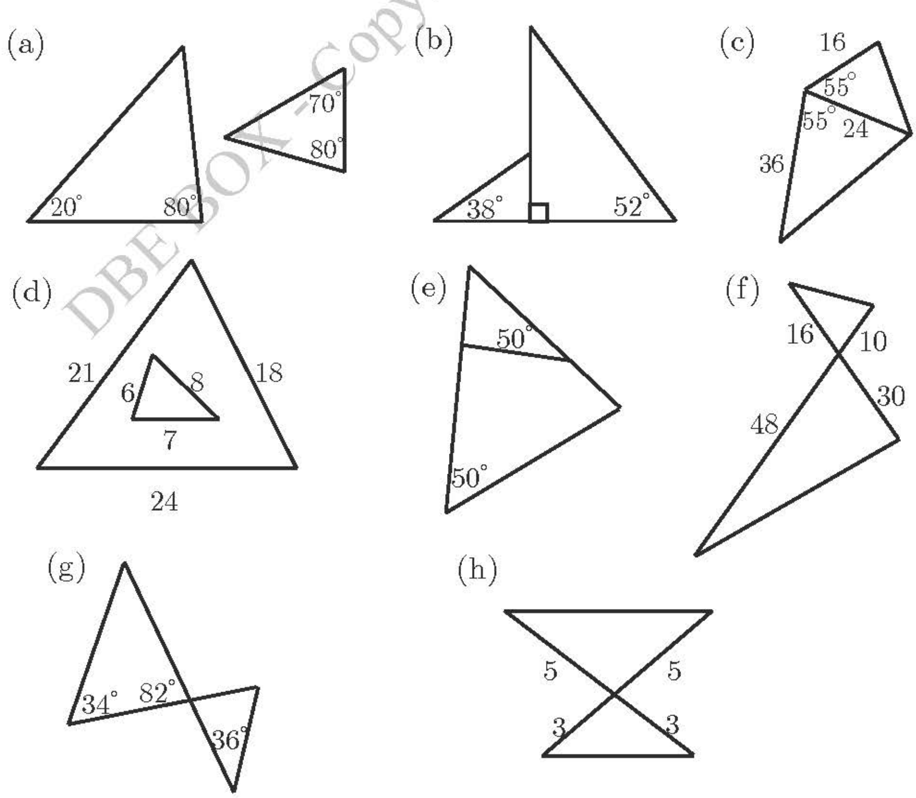
$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

$\triangle ABC \sim \triangle DEF (SSS)$

$\therefore \angle D = 80°.$

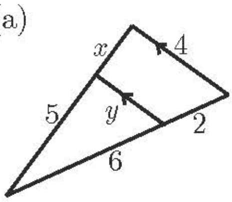
Exercise 8.3

1. Use the given information to tell whether each pair of triangles is similar or not. Give a reason for each answer.

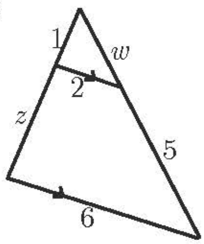


2. In each of the following triangles, the lengths of certain segments are marked. Find the values of x, y, z, w and v.

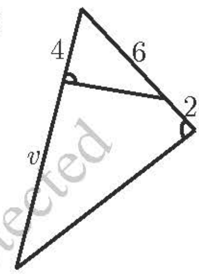
(a)



(b)

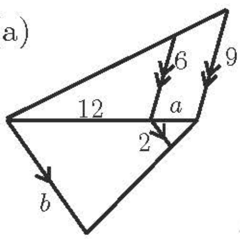


(c)

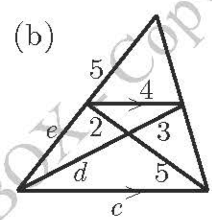


3. Find the marked lengths in each of the figures.

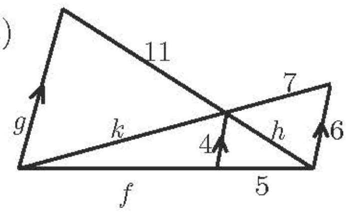
(a)



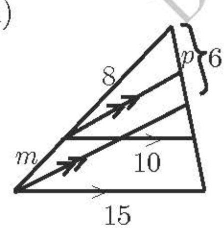
(b)



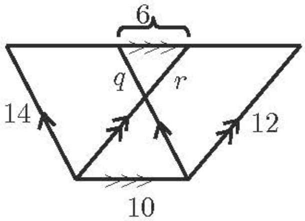
(c)



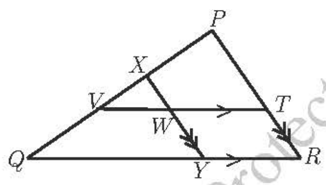
(d)



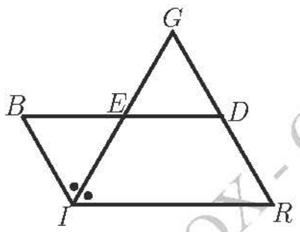
(e)



4. In the figure, $XY \parallel PR$ and $VT \parallel QR$. If $\frac{PT}{TR} = \frac{3}{2}$, $\frac{QY}{YR} = \frac{2}{1}$ and $PQ = 15$ cm, calculate
   (a) the lengths of PV, PX and XV.
   (b) the numerical values of $\frac{YW}{WX}$ and $\frac{VW}{QY}$.



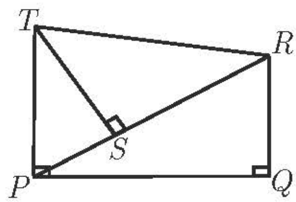
5.



Given : Parallelogram $BIRD$;
IG bisects $\angle BIR$.

Prove : $\frac{BE}{EI} = \frac{RG}{GI}$.

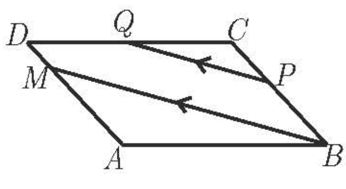
6.



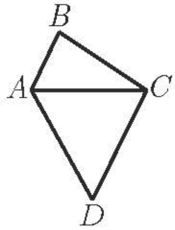
Given : $RQ \perp PQ, PQ \perp PT$,
$ST \perp PR$.

Prove : $ST \cdot RQ = PS \cdot PQ$.

7. Given : Parallelogram $ABCD$;
   $PQ \parallel MB$.
   Prove : $\triangle ABM \sim \triangle CQP$.



8. $\triangle ABC$ and $\triangle CAD$ are drawn on opposite sides of AC such that $AB : BC : CA = CA : AD : DC$. Prove that $DC \parallel AB$.



8.4 The Angle Bisector Theorem

An idea to be used in this section is the notion of dividing a segment internally or externally in a given ratio.

If B is a point on the line containing segment AC, then $\frac{AB}{BC}$ is the ratio in which B divides AC.

(1)

$$\frac{AB}{BC} = \frac{2}{4} = \frac{1}{2}$$

In this case B divides AC internally in the ratio $1 : 2$.

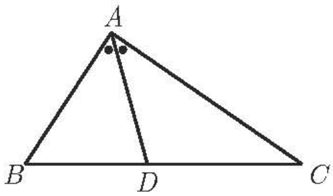
(2)

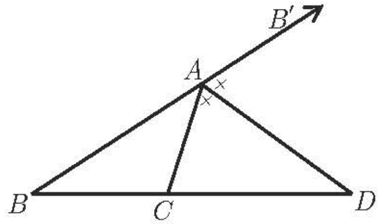
$$\frac{AB}{BC} = \frac{2}{4} = \frac{1}{2}$$

In this case B divides AC externally in the ratio $1 : 2$.

For a given segment there are usually two points which divide the segment in the given ratio. One is an internal point, the other is an external point. This is shown in above figures.

Theorem 5 (The Angle Bisector Theorem - ABT). The bisector of an interior(exterior) angle of a triangle divides the opposite side internally(externally) into a ratio equal to the ratio of the other two sides of the triangle.





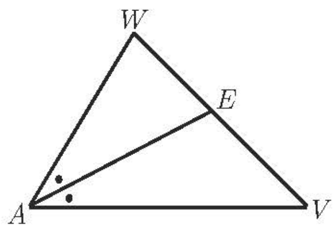
In $\triangle ABC$, if AD bisects $\angle BAC\,(\angle B'AC)$ then $\frac{AB}{AC} = \frac{BD}{DC}$.

Exercise 8.4

1. Which of the following proportions follow from the fact that AE bisects $\angle WAV$ in $\triangle WAV$?

(a) $\frac{WE}{EV} = \frac{WA}{AV}$ (b) $\frac{WE}{EV} = \frac{VA}{AW}$

(c) $\frac{WE}{WA} = \frac{EV}{AV}$ (d) $\frac{AV}{AW} = \frac{VE}{EW}$

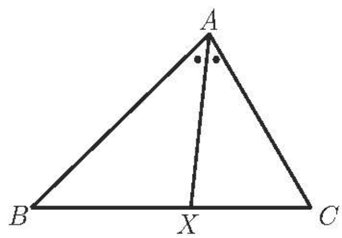


2. AX bisects $\angle CAB$. Complete the following statements:

(a) $AC : AB = ...$

(b) $AB : AC = ...$

(c) $XC : XB = ...$

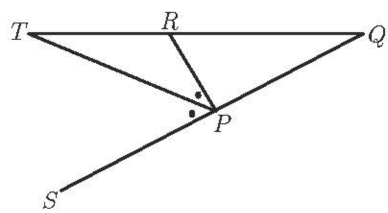


3. PT bisects $\angle RPS$. Complete the following statements:

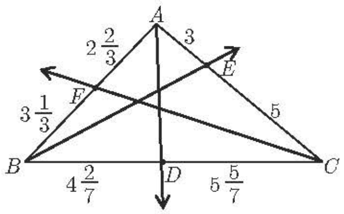
(a) $PQ : PR = ...$

(b) $TR : PR = ...$

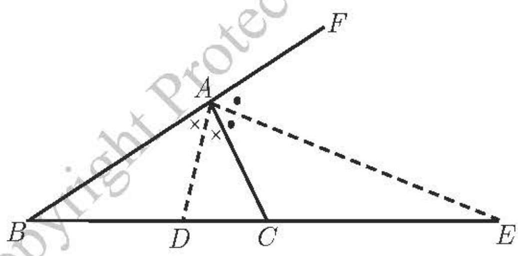
(c) $QR : TR = ...$



4. What can you say about the rays AD, BE and CF?

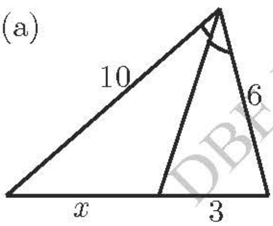


5. If AD and AE are bisectors of the interior and exterior angles at A of $\triangle ABC$, then which of the following are true?
   (a) $\angle DAE = 90°$
   (b) $BD : DC = BC : CE$
   (c) $BD : DC = BE : CE$
   (d) $AD : AE = DC : CE$

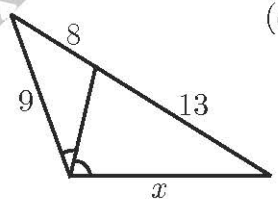


6. Find the value of x in each of the following figures.

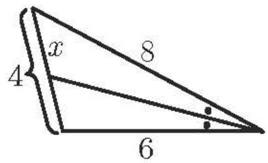
(a)



(b)



(c)



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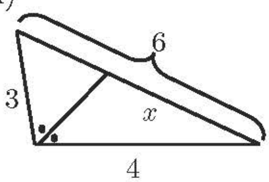
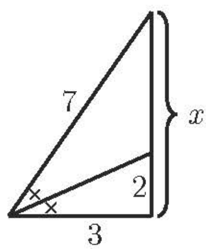
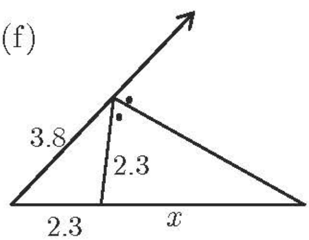


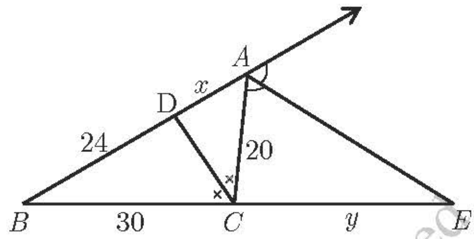
(e)



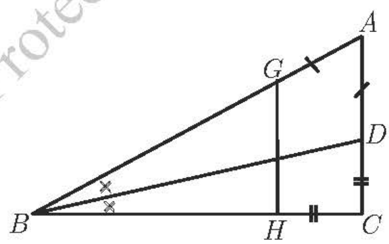
(f)



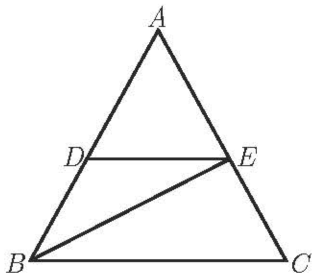
7. Find the unknown marked lengths in the figure.



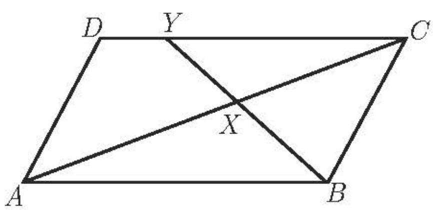
8. $AB = 12$ cm, $BC = 9$ cm, $CA = 7$ cm. BD bisects $\angle B$ and $AG = AD$, $CH = CD$. Calculate BG, BH. Does $GH \parallel AC$?



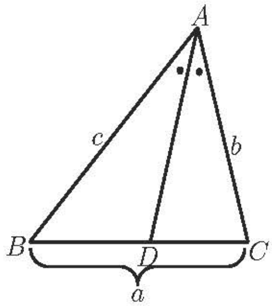
9. In $\triangle ABC, DE \parallel BC, AD = 2.7$ cm. $DB = 1.8$ cm and $BC = 3$ cm. Prove that BE bisects $\angle ABC$.



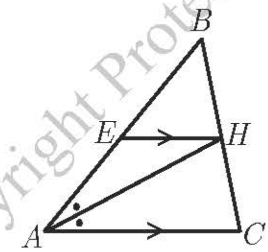
10. In a parallelogram $ABCD, AB = 3.6$ cm, $BC = 2.7$ cm. $AX = 3.2$ cm, $XC = 2.4$ cm. Prove that $\triangle BCY$ is isosceles.



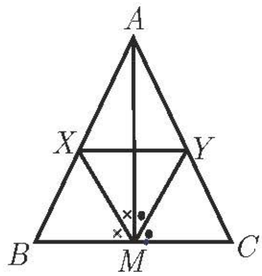
11. Calculate BD and DC in terms of a, b, c.



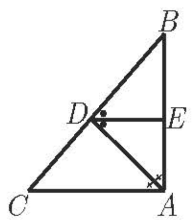
12. Given : AH bisects $\angle BAC$ in $\triangle ABC$.
$EH \parallel AC$.
Prove : $\frac{BE}{EA} = \frac{BA}{AC}$



13. Given : In $\triangle ABC, BM = MC$;
MX bisects $\angle AMB$
MY bisects $\angle AMC$.
Prove : $XY \parallel BC$.



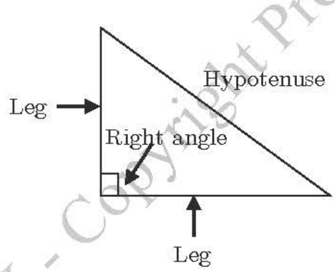
14. Given : In $\triangle ABC, \angle A = 2\angle C$,
AD bisects $\angle BAC$ and
DE bisects $\angle ADB$.
Prove : $\frac{BE}{EA} = \frac{BA}{AC}$.



8.5 The Pythagoras Theorem

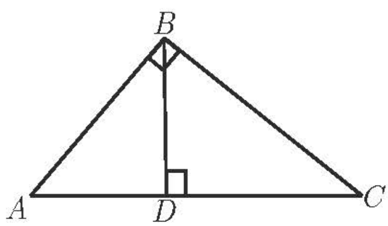
With our knowledge of similar triangles, we can prove the most famous theorem in Geometry, which is attributed to the Greek mathematician Pythagoras, who lived in the 6th century $B.C$. This theorem gives a relationship between the three sides of a right triangle or right-angled triangle. The first proof of this theorem is usually attributed to the Pythagoreans (a sect founded by Pythagoras).

Recall the definition that a triangle with a right angle is called a right triangle and the sides which determine the right angle are called legs of the right triangle, and the side opposite the right angle is called the hypotenuse.



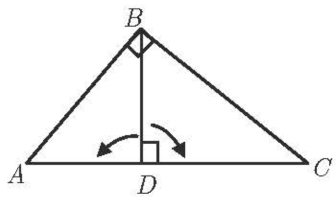
Theorem 6. The altitude to the hypotenuse of right triangle forms two triangles that are similar to each other and to the original triangle.

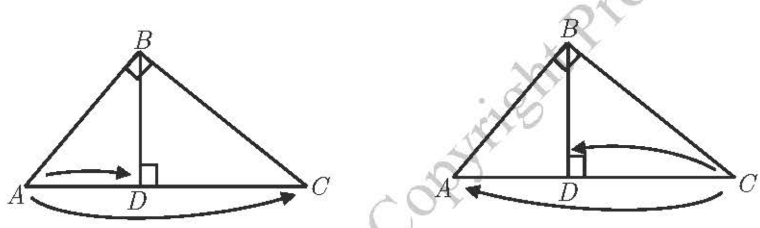
In a triangle ABC with $\angle ABC = 90°$, if BD is an altitude then $\triangle ADB \sim \triangle BDC \sim \triangle ABC$.



Corollary 6.1. The altitude to the hypotenuse of a right triangle is the **geometric mean** of the segments into which it separates the hypotenuse, and each leg of a right triangle is the geometric mean of the hypotenuse and the segment of the hypotenuse adjacent to the leg.

In a triangle ABC with $\angle ABC = 90°$, if BD is an altitude then
(1) $BD^2 = AD \cdot DC$
(2) $AB^2 = AD \cdot AC$
(3) $BC^2 = CD \cdot CA$.





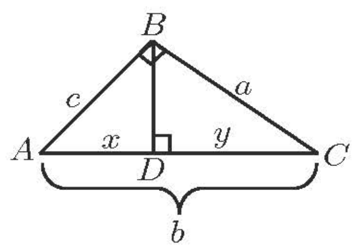
Now we will state and prove the most famous theorem in geometry.

Theorem 7 (Pythagoras Theorem). In any right triangle, the square of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

Given : $\triangle ABC$ is a right triangle with $\angle B = 90°$
To Prove : $b^2 = c^2 + a^2$
Proof : Draw $BD \perp AC$.
Then $\triangle ADB \sim \triangle BDC \sim \triangle ABC$ (Theorem 6)
$c^2 = xb$ and $a^2 = yb$ (by Corollary 6.1)
By addition,

$$xb + yb = c^2 + a^2$$
$$b(x + y) = c^2 + a^2$$

Since $x + y = b$, we get $b(b) = c^2 + a^2$.
Thus $b^2 = c^2 + a^2$.



The converse of the Pythagoras Theorem provides a way of showing whether or not a triangle is a right triangle. It is stated here without proof.

Converse of Pythagoras Theorem: If a triangle has sides with lengths a, b, c and $a^2 + b^2 = c^2$, then the triangle is a right triangle.

8.6 Special Right Triangles

There are two special types of right triangles that are of particular interest. One is the isosceles right triangle; such a triangle is formed by two sides and a diagonal of a square (Fig. 8.1(a)). The other type is the right triangle with acute angles of measures 30° and 60° ; an altitude of an equilateral triangle determines two such triangles. (Fig. 8.1(b))

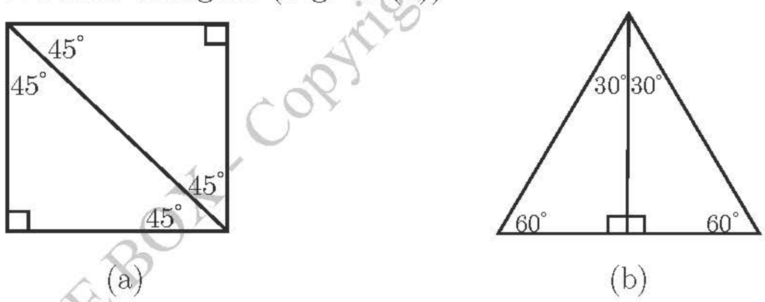
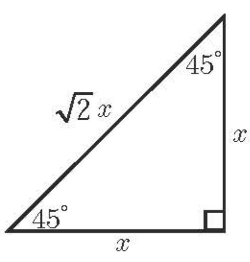


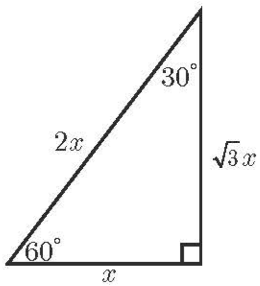
Fig. 8.1

The following theorems are based on Pythagoras Theorem, and therefore their proofs are left as exercises. There are frequent opportunities in geometry to apply these two theorems.

Theorem 8. In a 45°-45° right triangle, the length of hypotenuse is equal to the length of each leg times $\sqrt{2}$.

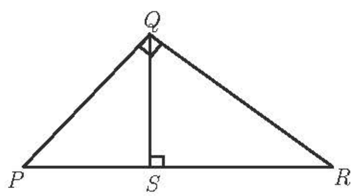


Theorem 9. In a 30°-60° right triangle, the leg opposite the 30° angle is one-half the length of the hypotenuse, and the other leg is equal to the length of hypotenuse times $\frac{\sqrt{3}}{2}$.

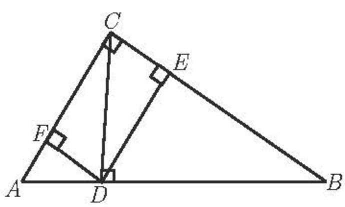


Exercise 8.5

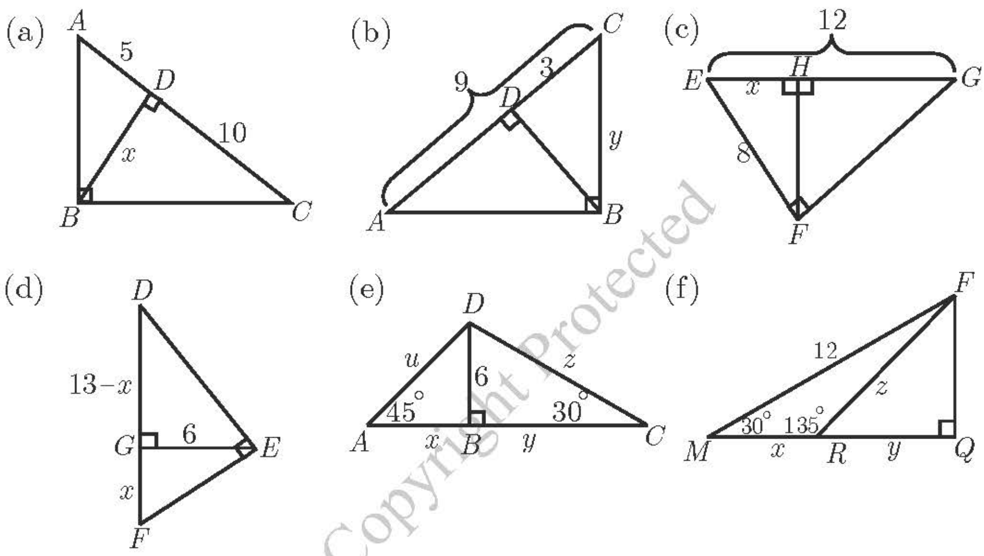
1. In the figure $\angle PQR = 90°, QS \perp PR$. Complete each of the following true statements.
   (a) $\triangle PQR \sim \triangle ? \sim \triangle ?$
   (b) QS is the geometric mean between ? and ?
   (c) QR is the geometric mean between ? and ?
   (d) $\frac{?}{PQ} = \frac{PQ}{?}$



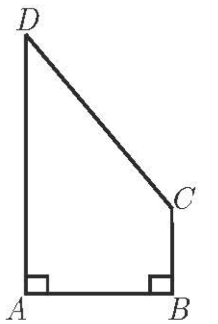
2. In the figure $CD \perp AB$ and $\angle C = 90°$. If $DE \perp BC, DF \perp CA$, write out all the triangles that are similar to $\triangle ABC$.



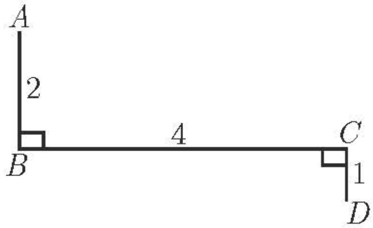
3. Find the length of each marked segment.



4. In the figure, if $AD = 10$ cm, $AB = 8$cm, $BC = 4$ cm, find the length of CD.

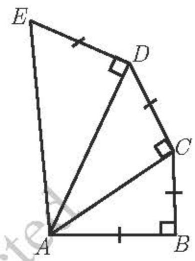


5. In the figure, find the distance of D from A.

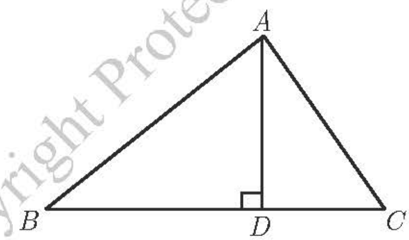


6. A parallelogram with sides 8 cm and 15 cm has a diagonal of 17 cm. Is it a rectangle?

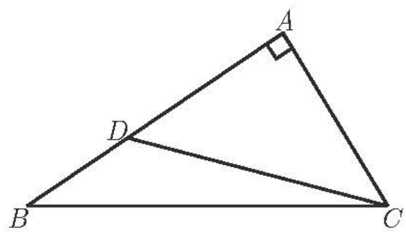
7. In the figure, $\triangle ABC, \triangle ACD, \triangle ADE$ are right triangles and $AB = BC = CD = DE$. Show that $AE = 2AB$.



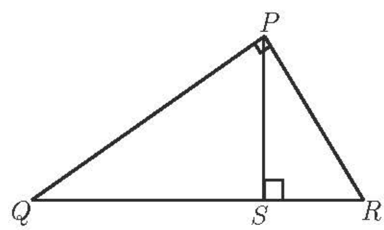
8. Given : $AD \perp BC$
   Prove : $AB^2 - AC^2 = BD^2 - DC^2$



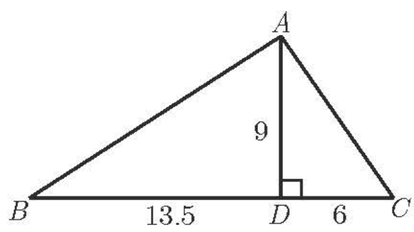
9. Given : $\angle BAC = 90°$
   D is any point on AB.
   Prove : $BC^2 + AD^2 = AB^2 + CD^2$



10. Given : $\angle QPR = 90°, PS \perp QR$.
    Prove : $\dfrac{1}{PS^2} = \dfrac{1}{PQ^2} + \dfrac{1}{PR^2}$



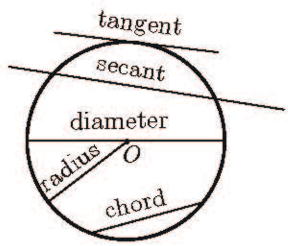
11. Given : $AD \perp BC, AD = 9$ cm,
    $BD = 13.5$ cm, $DC = 6$ cm.
    Prove : $\angle BAC = 90°$.



Chapter 9

Circles

In this chapter, properties of angles in a circle and properties of chords will be studied. We recall basic terms of a circle.

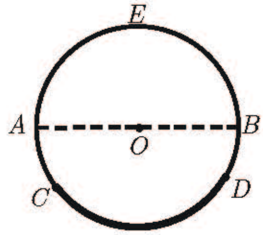


A **circle** is the set of all points that are at a fixed distance from a fixed point. The fixed point producing a circle is called the **centre** of the circle. A circle with centre O is called **circle O** and denoted by $\odot O$. Circles having the same centre are called **concentric circles**.

A **radius** is a segment joining the centre and a point on the circle. A segment joining two points on a circle is called a **chord** of the circle. A **diameter** is a chord passing through the centre of a circle. In a circle, a diameter is a longest chord. Circles having the same radius are called **congruent circles**.

A **secant** of a circle is a line that intersects the circle at two points. A line touching the circle at one point only is called a **tangent** to the circle. This touching point is called the **point of contact**.

An **arc** is a part of a circle. A **semicircle** is a half part of a circle. A **minor arc** is an arc shorter than a semicircle. A **major arc** is an arc longer than a semicircle.

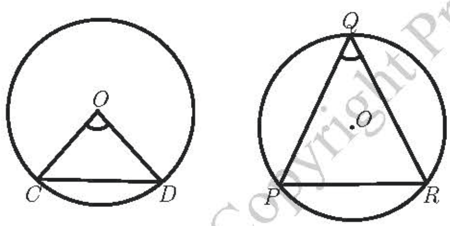


In the above figure, arc AEB is a semicircle, arc CD is a minor arc and arc CED is a major arc.

9.1 Angles in a Circle

Central Angles and Inscribed Angles

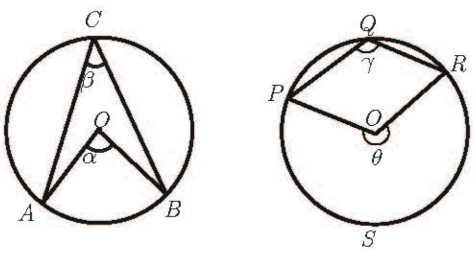
A **central angle** is an angle subtended by an arc (or a chord) of a circle at the centre. An **Inscribed angle** is an angle subtended by an arc (or a chord) of a circle at a point on the other arc.



In the above figure, $\angle COD$ is a central angle subtended by arc CD, and $\angle PQR$ is an inscribed angle subtended by arc PR.

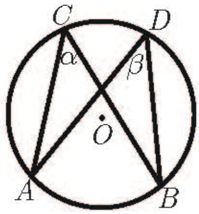
The following theorem and corollaries are properties of inscribed angles.

Theorem 1. The central angle is twice the inscribed angle subtended by the same arc.



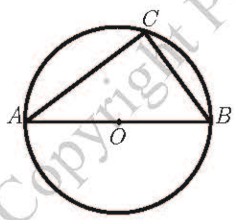
In $\odot O$, arc AB subtends central angle AOB and inscribed angle ACB. Then $\alpha = 2\beta$. Also, arc PSR subtends central angle POR and inscirbed angle PQR. Then $\theta = 2\gamma$.

Corollary 1.1. Inscribed angles subtended by the same arc are equal.



$\angle ACB$ and $\angle ADB$ are inscribed angles subtended by arc AB. Then $\alpha = \beta$.

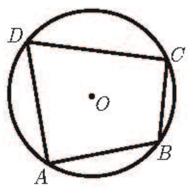
Corollary 1.2. An inscribed angle subtended by a diameter is a right angle.



In $\odot O$, AB is a diameter. Then $\angle ACB = 90°$.

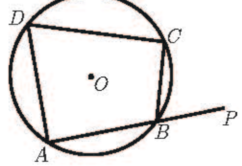
Cyclic Quadrilateral
A quadrilateral whose vertices lie on a circle is called a **cyclic quadrilateral**.

Theorem 2. Opposite angles of a cyclic quadrilateral are supplementary.



$ABCD$ is a cyclic quadrilateral. Then $\angle A + \angle C = 180°$ and $\angle B + \angle D = 180°$.

Corollary 2.1. The exterior angle of a cyclic quadrilateral is equal to the opposite interior angle of the quadrilateral.



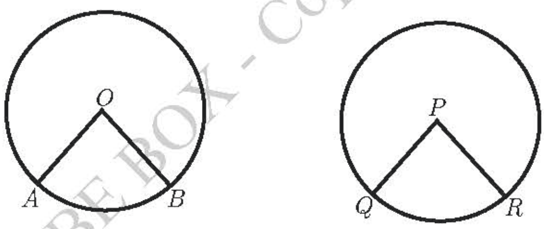
$ABCD$ is a cyclic quadrilateral. Then $\angle PBC = \angle D$.

The relations between central angles and arcs are stated in the following theorem.

Theorem 3. In the same circle or in congruent circles,

(i) arcs subtending equal central angles are equal,

(ii) equal arcs subtend equal central angles.



(i) Given: $\odot O$ and $\odot P$ are congruent and $\angle AOB = \angle QPR$.

To prove: arc AB = arc QR.

Proof: Place $\odot O$ on $\odot P$ so that O and P are coincide and OA falls along PQ.
Then $OA = PQ$. ($\because$ radii of congruent circles)
Hence A coincides with Q.
Since $\angle AOB = \angle QPR$, then OB falls along PR.
Since $OB = PR$, then B coincides with R.
Since every point on arc AB is equally distant from centre O and every point on arc QR is equally distant from centre P, then arc AB coincides with arc QR.
$\therefore$ arc AB = arc QR.

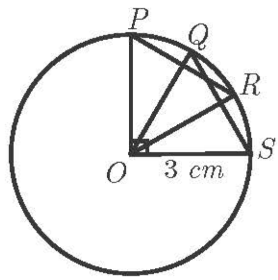
(ii) Given: $\odot O$ and $\odot P$ are congruent and arc AB = arc QR.
To prove : $\angle AOB = \angle QPR$.
Proof: Place $\odot O$ on $\odot P$ so that O and P are coincide and OA falls along PQ.
Then $OA = PQ$ ($\because$ radii of congruent circles)
Hence A coincides with Q.
Since arc AB = arc QR, then B coincides with R.
Therefore $\angle AOB = \angle QPR$.

By Theorem 1 and Theorem 3, we get the following theorem.

Theorem 4. In the same circle or in congruent circles, two inscribed angles are equal if and only if the corresponding arcs are equal.

Example 1.

In the given figure, O is the centre of the circle, arc PQ = arc QR = arc RS. Find PR.



Solution

arc PQ = arc QR = arc RS (given)

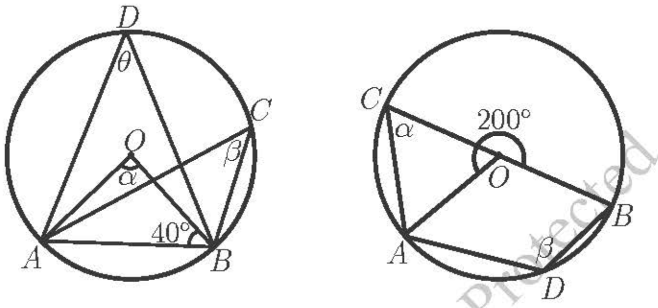
$$\begin{aligned}
\therefore \angle POQ &= \angle QOR = \angle ROS \\
\text{But } \angle POQ &+ \angle QOR + \angle ROS = 90^\circ \\
\therefore \angle POQ &= \angle QOR = \angle ROS = 30^\circ \\
\angle POR &= \angle POQ + \angle QOR = 60^\circ \\
\angle P &= \angle R \qquad (\because OR = OP = \text{radius}) \\
\therefore \angle P &= \angle R = \frac{180^\circ - 60^\circ}{2} = 60^\circ
\end{aligned}$$

So $\triangle POR$ is equilateral.
$\therefore PR = OR$

But $OR = OS$ (radii)
$\therefore PR = OS = 3$ cm

Example 2.

In $\odot O$, find the values of α, β and θ.



(a) (b)

Solution

(a) $\angle OAB = \angle OBA$ (radii)

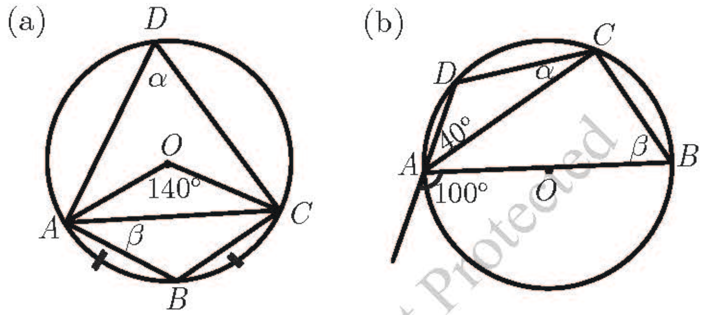
$\therefore \angle OAB = 40°$

$$\begin{aligned} \therefore \alpha &= 180° - (40° + 40°) = 100° \\ \beta &= \frac{1}{2}\alpha \quad \text{(subtended by arc } AB) \\ \therefore \beta &= 50° \\ \theta &= \beta \quad \text{(subtended by arc} AB) \\ \therefore \theta &= 50°. \end{aligned}$$

(b)
$$\begin{aligned} \beta &= \frac{1}{2} \times 200° \quad \text{(subtended by arc } ACB) \\ \therefore \beta &= 100° \\ \alpha + \beta &= 180° \quad \text{(opposite angles of cyclic quadrilateral } ACBD) \\ \alpha &= 180° - 100° = 80°. \end{aligned}$$

Example 3.

In the diagram below, O is the centre of the circle. Find the values of α and β.



Solution

(a) $\alpha = \frac{1}{2}\angle AOC$ (subtended by arc ABC)

$\alpha = 70°$

$\angle B + \alpha = 180°$ (opposite angles of cyclic quadrilateral $ABCD$)

$\therefore \angle B = 180° - \alpha = 110°$

$\beta = \angle BCA$ (arcAB = arcBC)

But $\beta + \angle B + \angle BCA = 180°$

$\therefore \beta = \frac{180° - \angle B}{2} = \frac{180° - 110°}{2} = 35°$

(b) $\angle ACB = 90°$ (angle subtended by diameter AB)

$\alpha + \angle ACB = 100°$ (exterior angle and opposite interior angle)

$\alpha = 100° - 90° = 10°$

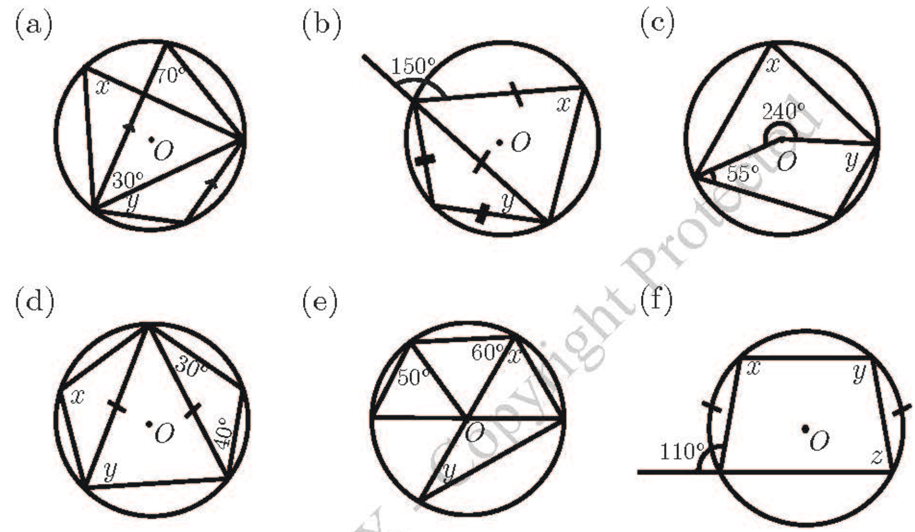
$\angle D + \angle DAC + \alpha = 180°$

$\therefore \angle D = 180° - (40° + \alpha) = 130°$

$\angle D + \beta = 180°$ (opposite angles of cyclic quadrilateral $ABCD$)
$\beta = 180° - \angle D = 50°$.

Exercise 9.1

1. The point O is the centre of the given circle. Find the values of x, y and z.



2. ABC is an acute triangle inscribed in $\odot O$, and OD is the perpendicular drawn O to BC. Prove that $\angle BOD = \angle BAC$.

3. In $\odot O$, two chords AB and CD intersect in the circle at P. Show that $\angle APD = \frac{1}{2}(\angle AOD + \angle BOC)$.

4. Two circles intersect at M, N. From M, diameters MA, MB are drawn in each circle. If A, B are joined to N, prove that ANB is a straight line.

5. OA and OB are two radii of a circle meeting at right angles. From A, B two parallel chords AX, BY are drawn. Prove that $AY \perp BX$.

6. Two circles intersect at R and S. Two straight lines ARB and CSD are drawn meeting one circle at A, C and the other at B, D. Prove that $AC \parallel BD$. If $AB \parallel CD$, show that $AB = CD$.

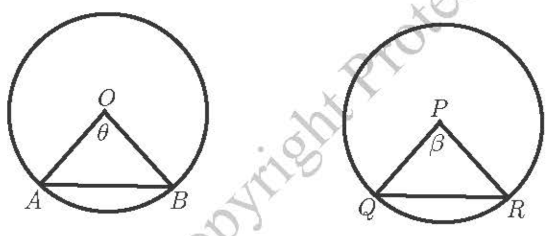
9.2 Properties of Chords

In this section, the relations between chords and central angles, symmetrical properties of chords, and properties of lengths of segments formed by chords and secants will be studied.

Theorem 5. In the same circle or in congruent circles,

(i) equal chords subtend equal central angles,

(ii) equal central angles cut off equal chords.

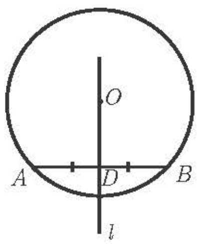


$\odot O$ and $\odot P$ are congruent. $AB = QR$ if and only if $\theta = \beta$.

From the above theorem, we can conclude that in the same circle or in congruent circles, two central angles are equal if and only if two corresponding minor arcs are equal if and only if two corresponding chords are equal.

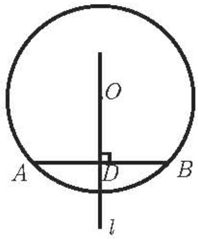
The following four theorems are the symmetric properties of chords.

Theorem 6. If a line passing through the centre of a circle bisects a chord of the circle, then the line is perpendicular to the chord.



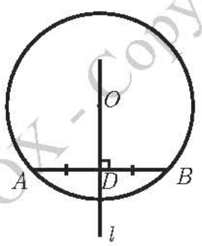
l is the line passing through the centre O and bisects chord AB at D. Then $l \perp AB$.

Theorem 7. If a line passing through the centre of a circle is perpendicular to a chord of the circle, then the line bisects the chord.



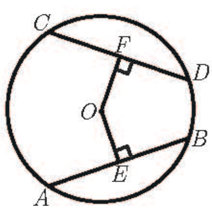
l is the line passing through the centre O and is perpendicular to chord AB at D. Then $AD = BD$.

Theorem 8. The perpendicular bisector of a chord of a circle passes through the centre.



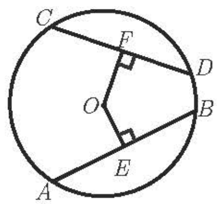
l is the perpendicular bisector of chord AB. Then l passes through the centre O.

Theorem 9. In the same circle or in congruent circles, chords are equal if and only if they are equidistant from the centre of the circle.



In $\odot O$, $AB = CD$ if and only if $OE = OF$.

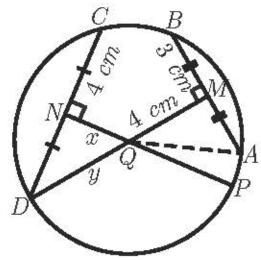
Theorem 10. Of any two chords of a circle, the greater chord is nearer to the centre, and conversely, the chord nearer to the centre is larger.



In $\odot O$, $AB > CD$ if and only if $OE < OF$.

Example 4.

In the given figure, find the values of x and y.



Solution

DM is the perpendicular bisector of AB.
Therefore the centre is on DM.
PN is the perpencicular bisector of CD.
Therefore the centre is also on PN.
DM and PN intersect at Q.
$\therefore Q$ is the centre of the given circle.
Join AQ.

$$AM = MB \quad \text{(given)}$$
$$\therefore AM = 3 \text{ cm}$$

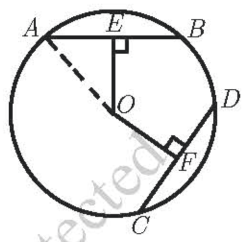
$$AQ^2 = AM^2 + MQ^2 = 3^2 + 4^2 = 25$$
$$\therefore AQ = 5 \text{ cm}$$
$$y = AQ \quad \text{(radii)}$$
$$\therefore y = 5 \text{ cm}$$

$$DN = NC \quad \text{(given)}$$
$$\therefore DN = 4 \text{ cm}$$
$$y^2 = x^2 + DN^2$$
$$\therefore x = \sqrt{y^2 - 4^2} = \sqrt{5^2 - 4^2} = 3 \text{ cm.}$$

Example 5.

Given $\odot O$, $OE = OF = 6$, $AE = x + 2$ and $CD = 3x - 2$. Find the radius of the circle.

Solution



$OE \perp AB$

$\therefore AB = 2AE = 2(x+2) = 2x + 4$

Since $OE = OF$, then $AB = CD$.

$\therefore 2x + 4 = 3x - 2$

$x = 6.$

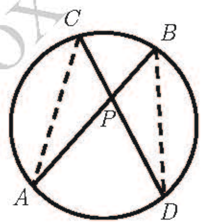
$\therefore AE = 6 + 2 = 8$

Join OA.

$\therefore$ radius $= OA = \sqrt{OE^2 + AE^2} = \sqrt{6^2 + 8^2} = 10.$

When two chords intersect each other inside or outside the circle, the product properties of the lengths of these line segments are as follows:

Theorem 11. If two chords of a circle intersect in the circle, the product of the lengths of segments of one chord is equal to the product of the lengths of segments of the other chord.



Given: Chord AB and chord CD intersect at a point P in the circle.

To prove: $PA \cdot PB = PC \cdot PD$

Proof: Join AC, BD.

In $\triangle APC$ and $\triangle DPB$,

$\angle A = \angle D$ (subtended by arc BC)

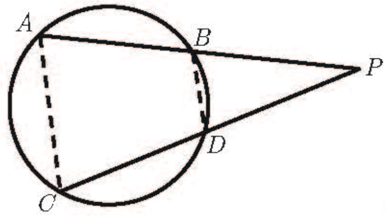
$\angle C = \angle B$ (subtended by arc AD)

$\therefore \triangle APC \sim \triangle DPB$ (AA Corollary)

$$\therefore \frac{PA}{PD} = \frac{PC}{PB}$$

$\therefore PA \cdot PB = PC \cdot PD.$

Theorem 12. If two secants are drawn to a circle from an external point, the product of the lengths of one secant and its external segment is equal to the product of the lengths of the other secant and its external segment.



Given: PBA and PDC are two secant segments.

To prove: $PA \cdot PB = PC \cdot PD$

Proof: Join AC, BD.

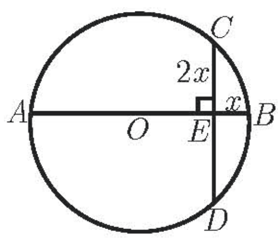
In $\triangle DPB$ and $\triangle APC$,

$$\begin{aligned} \angle P &= \angle P \\ \angle PDB &= \angle PAC \quad \text{(exterior and opposite interior)} \\ \therefore \triangle DPB &\sim \triangle APC \quad \text{(AA Corollary)} \\ \therefore \frac{PD}{PA} &= \frac{PB}{PC} \\ \therefore PA \cdot PB &= PC \cdot PD. \end{aligned}$$

Example 6.

Given $\odot O$, $AE = 12$, $BE = x$ and $CE = 2x$. Find radius of the circle.

Solution



$OB \perp CD$

$\therefore ED = CE = 2x$

AB and CD intersect at E.

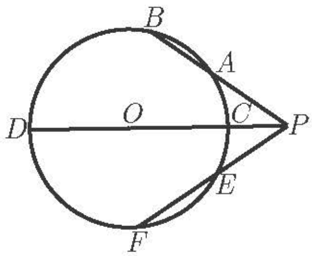
$$\begin{aligned} \therefore AE \cdot EB &= CE \cdot ED \\ 12 \cdot x &= 2x \cdot 2x \\ x &= 3 \end{aligned}$$

$AB = AE + EB = 12 + x = 15$

$\therefore$ radius $= OA = \frac{1}{2}AB = 7.5.$

Example 7.

Given $\odot O$, radius $= 6$, $PC = 3$, $PA = 5$ and $EF = 6$. Find AB and PE.



Solution

DC = diameter = 12
$PD = PC + CD = 3 + 12 = 15$

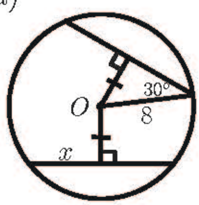
$$\begin{aligned} PA \cdot PB &= PC \cdot PD \\ 5(5 + AB) &= 3 \times 15 \\ 5 + AB &= 9 \\ AB &= 4. \end{aligned}$$

$$\begin{aligned} PE \cdot PF &= PC \cdot PD \\ PE(PE + 6) &= 3 \times 15 \\ PE^2 + 6PE - 45 &= 0 \\ PE^2 + 6PE + 9 &= 45 + 9 \\ (PE + 3)^2 &= 54 \\ PE + 3 &= \sqrt{54} \\ \therefore PE &= \sqrt{54} - 3 = 3\sqrt{6} - 3. \end{aligned}$$

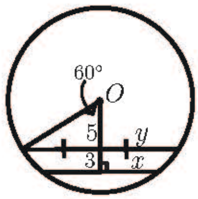
Exercise 9.2

1. In the following figures, O is the centre of circles. Find the values of x and y.

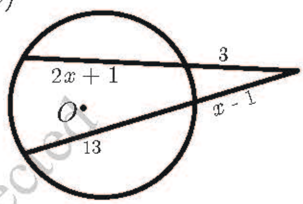
(a)



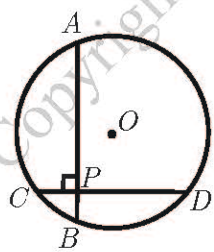
(b)



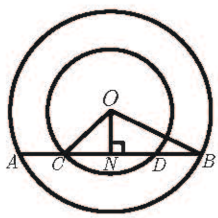
(c)



2. In $\odot O$, chords AB is perpendicular to CD at P, $AB = 16$, $CP = 4$, $PD = 10$. Find the radius.



3. In the figure, O is the centre of the concentric circles and $ON \perp AB$. If $OC = 10$, $ON = 8$ and $OB = 17$, find AC.



4. Prove Theorem 10: Of any two chords of a circle, the greater chord is nearer to the centre, and conversely, the chord nearer to the centre is larger.

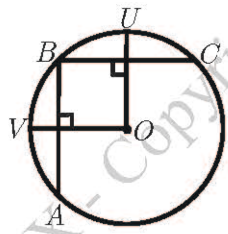
5. Let P be a point inside a circle. AB is the diameter passes through P and CPD is the chord perpendicular to AB. Show that CD is the shortest of all chords passing through P.

6. Through a point P in a circle, the longest chord that can be drawn is 10 cm long and the shortest chord is 6 cm long. What is the radius of the circle and how far is P from the centre?

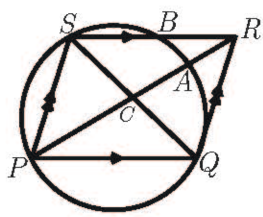
7. In $\odot O$, chords AB and CD are equal and intersect in the circle at E such that $AE < EB$ and $CE < ED$. Show that $\triangle BDE$ is isosceles with base BD.

8. In $\odot O$, congruent chords AB and CD are produced to meet at P. Prove that $\triangle PAC$ is isosceles.

9. In $\odot O$, AB and BC are equal chords, $OV \perp AB$, and $OU \perp BC$. Prove that B is the midpoint of arc VU.



10. In parallelogram $PQRS$, $PQ = 5$ cm, $PR = 8$ cm, $QS = 6$ cm. Calculate the lengths of AR and BR.



11. Chords AB and CD intersect at E and $AE = EB$. A semicircle is drawn with diameter CD. EF, perpendicular to CD, meets this semicircle at F. Prove that $AE = EF$.

12. $\odot O$ and $\odot P$ intersect at A and B. Show that OP is the perpendicular bisector of the common chord AB.

Chapter 10

Trigonometry

The word **trigonometry** is derived from the words "tri"(meaning three), "gon"(meaning sides) and "metry"(meaning measure). Thus trigonometry deals with the measurement of sides and angles of a triangle.

It has been widely used in Astronomy, Surveying, Geography, Physics, Navigation etc. The captain of a ship employs trigonometry to calculate the distance from the far off island, sea shores, cliffs and other ships on the high seas.

To study trigonometry, the students should already be acquainted with the theorems on similar triangles. Only then will they find it easy to understand the definition of trigonometric ratios. To begin with, we shall only consider the acute angles in this chapter. There is more to learn in trigonometry than measuring triangles. In fact, trigonometry generally deals with angles of all sizes with measurement not necessarily confined to the angles of triangles. In every branch of Higher Mathematics, whether Pure or Applied, a knowledge of trigonometry is of great value.

10.1 Angles

In trigonometry an angle is determined by rotating a ray about its endpoint from an initial position to terminal position.

Consider a line OP which is free to rotate in the XY-plane. O is taken to be the origin about which a line OP rotates.

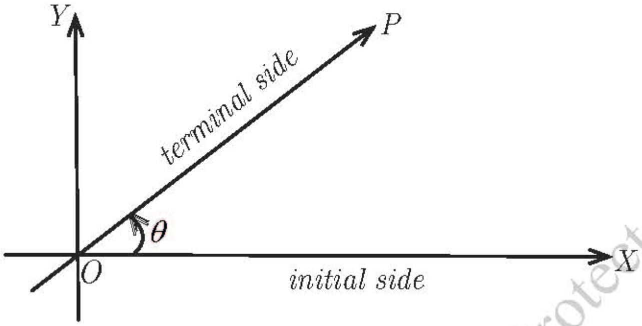


Figure 10.1

When the line OP is rotated, it is possible to vary the size of the angle θ between OP and OX (see Fig. 10.1). Angles measured from the X-axis (i.e., OX) in an anticlockwise direction are positive angles. Angles measured from the X-axis in a clockwise direction are negative angles.

Therefore in Fig. 10.2, the angle α (i.e., $\angle P_1OX$) is positive while the angle β (i.e., $\angle P_2OX$) is negative.

One complete revolution of the line OP from OX makes an angle of 360°. The range of values of θ between 0° and 360° in each quadrant is as shown in Fig. 10.3.

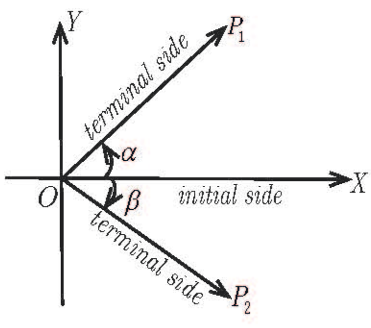


Figure 10.2

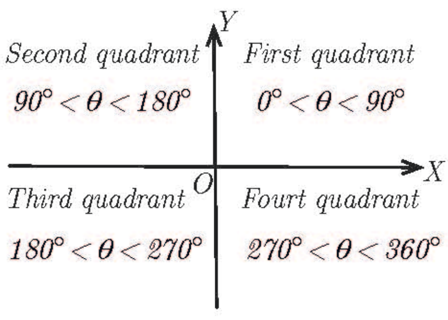


Figure 10.3

10.2 The Relation between Degree and Radian Measure

Two kinds of units commonly used for measuring angles are radian measure and degree measure. The radian measure is employed almost exclusively in advanced mathematics and in many branches of science. In this chapter first we introduce the concept of radian and study the relation between degrees and radians.

Consider a circle with centre O and radius r units as shown in Fig. 10.4. Let arc AB be an arc on the circle of length equal to r. We define the magnitude of angle AOB which the arc AB subtends at the centre as one radian.

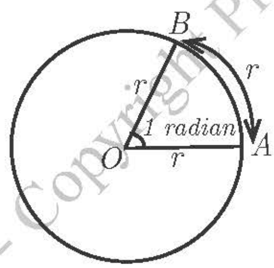


Figure 10.4

Since the circumference of a circle is equal to $2\pi r$, it subtends a central angle of 2π radians. That is there are 2π radians in a complete rotation of $360°$. Therefore 2π radians $= 360°$ and hence π radians $= 180°$ which is a fundamental relation between radians and degrees.

We have

$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees} \approx 57°19'$$

$$1° = \frac{\pi}{180} \text{ radians} \approx 0.01764 \text{ radians.}$$

Example 1.

Express the following in radian measures.

(a)$60°$ (b)$135°$

Solution

(a) $60° = 60 \times \frac{\pi}{180}$ radians $= \frac{\pi}{3}$ radians

(b) $135° = 135 \times \frac{\pi}{180}$ radians $= \frac{3\pi}{4}$ radians.

Example 2.

Express the following in degree measures.

(a) $\frac{\pi}{4}$ radians (b) $\frac{4\pi}{5}$ radians

Solution

(a) $\frac{\pi}{4}$ radians $= \frac{\pi}{4} \times \frac{180}{\pi}$ degrees $= 45°$

(b) $\frac{4\pi}{5}$ radians $= \frac{4\pi}{5} \times \frac{180}{\pi}$ degrees $= 144°$.

Note: Usually when the units of an angle are not specified, it is understood that the angle is expressed in radians.

10.3 Arc Length and Area of a Sector of a Circle

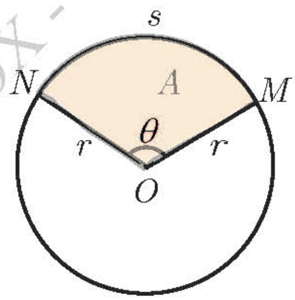


Figure 10.5

Let the arc MN subtend an angle of magnitude θ radians at the centre of a circle of radius r as shown in Fig. 10.5. Clearly the length "s" of arc MN is proportional to the angle θ and we have

$$\frac{\text{length of arc } MN}{\text{length of the circumference}} = \frac{\text{angle subtended by arc } MN}{\text{angle subtended by circumference}}$$

i.e., $$\frac{s}{2\pi r} = \frac{\theta}{2\pi} \quad \text{(or)} \quad \theta(\text{in radians}) = \frac{s}{r}$$

Furthermore, as the area of the sector MON (the shaded region shown in Fig. 10.5) is also, proportional to the angle θ, we have

$$\frac{\text{area of sector } MON}{\text{area of circle}} = \frac{\theta}{2\pi}$$

i.e.

$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$

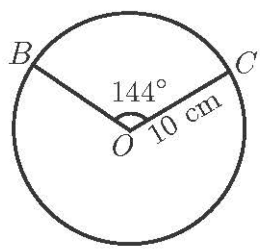
(or)

$$\boxed{A = \frac{1}{2}r^2\theta}$$

where θ is given in radian measures.

Example 3.

An arc BC subtends an angle of 144° at the centre O of a circle of radius 10cm. Find the length of arc BC and the area of the sector BOC.



Solution

$\theta = 144° = 144 \times \frac{\pi}{180} = \frac{4\pi}{5}$ radians and $r = 10$ cm.

The length of arc $BC = r\theta = 10 \times \frac{4\pi}{5} = 8\pi$ cm.

The area of the sector $BOC = \frac{1}{2}r^2\theta = \frac{1}{2} \times 10^2 \times \frac{4\pi}{5} = 40\pi\ cm^2$.

Exercise 10.1

1. Convert each of the following to radians.

   (a) $120°$ (b) $90°$ (c) $72°$ (d) $225°$

   (e) $150°$ (f) $108°$ (g) $160°$ (h) $390°$

2. Convert each of the following to degrees.

   (a) $\frac{\pi}{5}$ (b) $\frac{3\pi}{4}$ (c) $\frac{5\pi}{6}$ (d) π

   (e) $\frac{8\pi}{9}$ (f) $\frac{12\pi}{5}$ (g) $\frac{\pi}{3}$ (f) $\frac{7\pi}{3}$

3. A central angle θ subtends an arc of $\frac{11\pi}{2}$ cm on a circle of radius 6 cm. Find the measure of θ in radians and the area of a sector of a circle which has θ is its central angle.

4. The area of a sector of a circle is 143 cm^2 and the length of the arc of a sector is 11 cm. Find the radius of the circle.

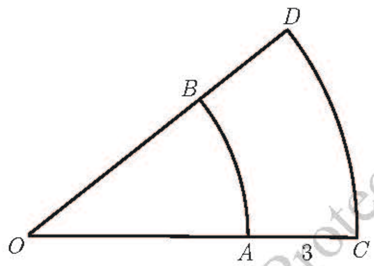
5. A sector cut from a circle of radius 3 cm has a perimeter of 16 cm. Find the area of this sector.

6. A piece of wire of fixed length L cm, is bent to form the boundary a sector of a circle. The circle has radius r cm and the angle of the sector is $\theta = (\frac{32}{r} - 2)$ radians. Find the wire of fixed length L and show that the area of the sector, A cm^2 is given by $A = 16r - r^2$.

7. A race is run at a uniform speed on a circular course. In each minute, a runner traverses an arc of a circle which subtends $2\frac{6}{7}$ radians at the centre of the course. If each lap is 792 yards, how long does the runner take to run a mile?

8. The large hand of a clock is 28 inches long; how many inches does its extremity move in 20 minutes?

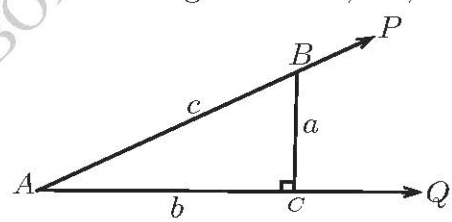
9. The figure shows two sectors in which the arcs AB and CD are arcs of concentric circles, centre O. If $\angle AOB = \frac{2}{3}$ radians, $AC = 3$ cm and the area of a sector AOB is 12 cm^2, calculate the area and the perimeter of $ABDC$.



10.4 Six Trigonometric Ratios

Consider the acute angle PAQ, that is an angle whose measure in degrees is less than 90°. On one arm AP of the angle PAQ, select a point B and draw BC perpendicular to AQ at C. Thus a right triangle ACB is formed.

Denote the lengths of the segments BC, AC, AB by the letters a, b, c



respectively.

We say that BC is the side which is *opposite* to angle A; AC is the side which is *adjacent* to angle A; AB is the *hypotenuse.* With reference to the angle A the following definitions are employed.

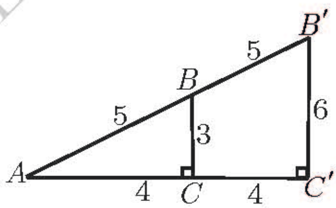
The ratio $\frac{BC}{AB}$ or $\frac{\text{opposite side of}\angle A}{\text{hypotenuse}}$ is called the **sine** of angle A (or $\sin A$).

The ratio $\frac{AC}{AB}$ or $\frac{\text{adjacent side of}\angle A}{\text{hypotenuse}}$ is called the **cosine** of angle A (or $\cos A$).

The ratio $\frac{BC}{AC}$ or $\frac{\text{opposite side of}\angle A}{\text{adjacent side of}\angle A}$ is called the **tangent** of angle A (or $\tan A$).
The ratio $\frac{AC}{BC}$ or $\frac{\text{adjacent side of}\angle A}{\text{opposite side of}\angle A}$ is called the **cotangent** of angle A (or $\cot A$).
The ratio $\frac{AB}{AC}$ or $\frac{\text{hypotenuse}}{\text{adjacent side of}\angle A}$ is called the **secant** of angle A (or $\sec A$).
The ratio $\frac{AB}{BC}$ or $\frac{\text{hypotenuse}}{\text{opposite side of}\angle A}$ is called the **cosecant** of angle A (or $\csc A$).

Since the measure of the angles are in degrees, then $\angle A = \alpha$ means that the measure of angle A is α degrees. Thus if $\angle A = \alpha$, then we may write $\sin A = \sin\alpha$, $\cos A = \cos\alpha$ and $\tan A = \tan\alpha$. Note the six ratios do not depend on the size of the triangle. The following example illustrates this.

Example 4.
(a) Using the right triangle ABC, find $\sin A$, $\cos A$, $\tan A$.
(b) Using the right triangle $AB'C'$, find $\sin A$, $\cos A$, $\tan A$.
Solution



(a) From the right triangle ABC,

$$\sin A = \frac{\text{opposite side of } \angle A}{\text{hypotenuse}} = \frac{BC}{AB} = \frac{3}{5},$$
$$\cos A = \frac{\text{adjacent side of } \angle A}{\text{hypotenuse}} = \frac{AC}{AB} = \frac{4}{5},$$
$$\tan A = \frac{\text{opposite side of } \angle A}{\text{adjacent side of } \angle A} = \frac{BC}{AC} = \frac{3}{4}.$$

(b) From the right triangle $AB'C'$,

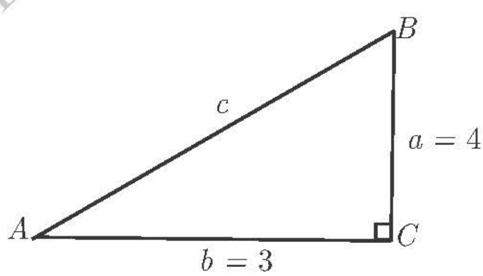
$$\begin{aligned}
\sin A &= \frac{\text{opposite side of } \angle A}{\text{hypotenuse}} = \frac{B'C'}{AB'} = \frac{6}{10} = \frac{3}{5}, \\
\cos A &= \frac{\text{adjacent side of } \angle A}{\text{hypotenuse}} = \frac{AC'}{AB'} = \frac{8}{10} = \frac{4}{5}, \\
\tan A &= \frac{\text{opposite side of } \angle A}{\text{adjacent side of } \angle A} = \frac{B'C'}{AC'} = \frac{6}{8} = \frac{3}{4}.
\end{aligned}$$

We see that $\frac{BC}{AB} = \frac{B'C'}{AB'} = \frac{3}{5}$, $\frac{AC}{AB} = \frac{AC'}{AB'} = \frac{4}{5}$, $\frac{BC}{AC} = \frac{B'C'}{AC'} = \frac{3}{4}$.

Thus $\sin A, \cos A$ and $\tan A$ do not depend on the size of the triangle.

Example 5.

ABC is a right triangle in which C is the right angle. If $a = 4$ and $b = 3$, find c, $\cos A$ and $\sec B$.



Solution

By Pythagoras theorem, $c^2 = a^2 + b^2 = 4^2 + 3^2 = 25$.

$$\therefore c = 5$$

$$\cos A = \frac{b}{c} = \frac{3}{5}$$

$$\sec B = \frac{c}{a} = \frac{5}{4}.$$

Example 6.

Given right triangle ABC with $\angle C = 90°$ and $\tan A = \frac{5}{12}$, find $\sin A$ and $\cos A$.

Solution

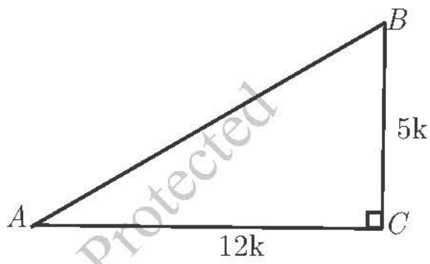
Since $\tan A = \frac{5}{12} = \frac{BC}{AC}$, we can take

$BC = 5k$ and $AC = 12k$, where k =constant.

By Pythagoras theorem,

$AB^2 = BC^2 + AC^2 = (5k)^2 + (12k)^2 = 169k^2$

$$\therefore AB = 13k.$$

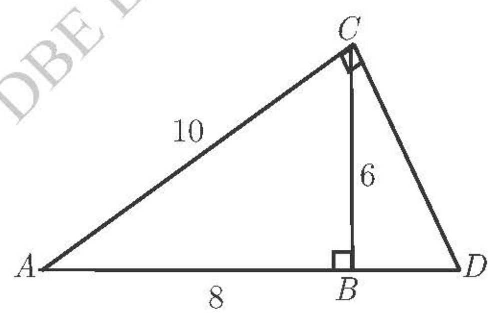


Hence, $\sin A = \frac{BC}{AB} = \frac{5k}{13k} = \frac{5}{13}$

$$\cos A = \frac{AC}{AB} = \frac{12k}{13k} = \frac{12}{13}.$$

Example 7.

Given right triangle ABC in which $\angle B = 90°$, CD is drawn perpendicular to CA and meets AB produced at D. If $BC = 6$, $AB = 8$, $AC = 10$, find CD and AD.



Solution

From the right triangle ABC, $\tan A = \frac{BC}{AB}$

From the right triangle ACD, $\tan A = \frac{CD}{AC}$

$$\therefore \frac{CD}{AC} = \frac{BC}{AB}$$

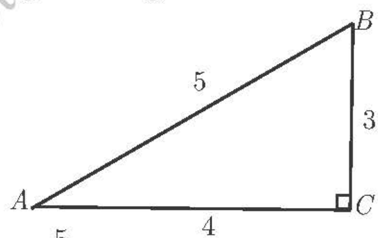
$$CD = \frac{BC}{AB} \times AC = \frac{6}{8} \times 10 = \frac{15}{2} = 7.5$$

$$\frac{AD}{AC} = \sec A = \frac{AC}{AB}$$

$$AD = \frac{AC^2}{AB} = \frac{10^2}{8} = \frac{100}{8} = \frac{25}{2} = 12.5.$$

Exercise 10.2

1. Find the following trigonometric ratios for a right triangle with sides as indicated,

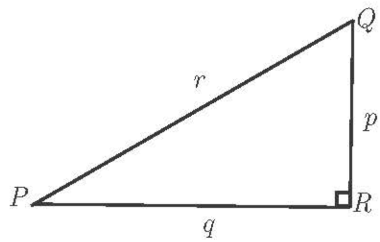


$\sin A$, $\cos A$, $\tan A$, $\cot A$, $\sec A$, $\csc A$,
$\sin B$, $\cos B$, $\tan B$, $\cot B$, $\sec B$, $\csc B$.

2. Given a right $\triangle ABC$ with $\angle C = 90°$ and $\cos A = \frac{5}{13}$, determine the values of $\tan A$ and $\sin A$.

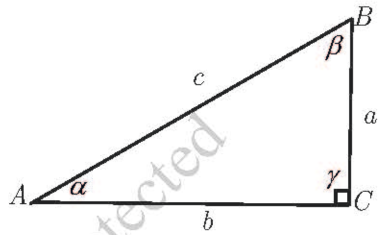
3. Given a right $\triangle PQR$ with $\angle R = 90°$; if $PQ = 13$, $QR = 5$, $PR = 12$, find the values of $\tan P$ and $\sec Q$, $\csc Q$, $\sin P$.

4. Calculate each of the following for a right $\triangle PQR$ with $\angle R = 90°$.
(i) $(\cos P)(\sec P)$ (ii) $(\tan Q)(\cot Q)$ (iii) $(\sin P)(\csc P)$
(iv) $\sec^2 Q - \tan^2 Q$ (v) $\sin^2 P + \cos^2 P$ (vi) $\csc^2 Q - \cot^2 Q$



5. $PQRS$ is a quadrilateral in which $\angle PSR = 90°$. If the diagonal PR is at right angles to RQ, and $RP = 20$, $RQ = 21$, $RS = 16$, find $\sin \angle PRS$, $\tan \angle RPS$, $\cos \angle RPQ$, and $\csc \angle PQR$.

10.5 Relations between the Trigonometric Ratios



Let $\triangle ABC$ be a right triangle with $\angle C = 90°$.

(i) Then $\sin A = \frac{a}{c}$, and $\csc A = \frac{c}{a}$.
$\therefore \sin A \times \csc A = \frac{a}{c} \times \frac{c}{a} = 1$.
Thus $\sin A$ and $\csc A$ are reciprocals.
$\therefore \sin A = \frac{1}{\csc A}$ and $\csc \text{A} = \frac{1}{\sin A}$.

(ii) $\cos A = \frac{b}{c}$, and $\sec A = \frac{c}{b}$.
$\therefore \cos A \times \sec A = \frac{b}{c} \times \frac{c}{b} = 1$.
Thus $\cos A$ and $\sec A$ are reciprocals.
$\therefore \cos A = \frac{1}{\sec A}$ and $\sec A = \frac{1}{\cos A}$.

(iii) $\tan A = \frac{a}{b}$, and $\cot A = \frac{b}{a}$.
$\therefore \tan A \times \cot A = \frac{a}{b} \times \frac{b}{a} = 1$.
Thus $\tan A$ and $\cot A$ are reciprocals.
$\therefore \tan A = \frac{1}{\cot A}$ and $\cot A = \frac{1}{\tan A}$.

(iv) $\tan A = \frac{a}{b} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{\sin A}{\cos A}$, and $\cot A = \frac{b}{a} = \frac{\frac{b}{c}}{\frac{a}{c}} = \frac{\cos A}{\sin A}$.

(v) $\sin^2 A + \cos^2 A = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = \frac{a^2 + b^2}{c^2} = \frac{c^2}{c^2} = 1$.
Similarly, we can prove that

$$\begin{aligned} 1 + \tan^2 A &= \sec^2 A \\ \text{and} \qquad 1 + \cot^2 A &= \csc^2 A. \end{aligned}$$

(vi) Let $\angle A = \alpha$, $\angle B = \beta$ and $\angle C = \gamma = 90°$.

From the figure, we see that $\alpha + \beta = 90°$.

$$\therefore \beta = 90° - \alpha.$$

$$\therefore \sin(90° - \alpha) = \sin\beta = \frac{b}{c} = \cos\alpha \text{ and}$$

$$\cos(90° - \alpha) = \cos\beta = \frac{a}{c} = \sin\alpha.$$

Similarly, we can prove that

$$\begin{aligned} \tan(90° - \alpha) &= \cot\alpha \\ \cot(90° - \alpha) &= \tan\alpha \\ \sec(90° - \alpha) &= \csc\alpha \\ \csc(90° - \alpha) &= \sec\alpha. \end{aligned}$$

Example 8.

Prove that $\dfrac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \tan\theta$.

Solution

$$\begin{aligned} \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} &= \frac{\sin\theta}{\sqrt{\cos^2\theta}} \quad (\because \sin^2\theta + \cos^2\theta = 1) \\ &= \frac{\sin\theta}{\cos\theta} \\ &= \tan\theta. \end{aligned}$$

Example 9.

Verify the identity $(1-\sin\theta)(1+\sin\theta) = \dfrac{1}{1+\tan^2\theta}$.

Solution

$$\begin{aligned} (1-\sin\theta)(1+\sin\theta) &= 1-\sin^2\theta \\ &= \cos^2\theta \quad (\because \sin^2\theta + \cos^2\theta = 1) \\ &= \frac{1}{\sec^2\theta} \quad (\because \cos^2\theta = \frac{1}{\sec^2\theta}) \\ &= \frac{1}{1+\tan^2\theta}. \end{aligned}$$

Example 10.

Find the value of acute angle α when $\cos 3\alpha = \sin 2\alpha$.
Solution

$$\begin{aligned} \cos 3\alpha &= \sin 2\alpha \\ \sin(90^\circ - 3\alpha) &= \sin 2\alpha \\ \therefore 90^\circ - 3\alpha &= 2\alpha \\ 5\alpha &= 90^\circ \\ \alpha &= 18^\circ. \end{aligned}$$

Exercise 10.3

Prove the following identities.

1. $\cot\theta\sqrt{1-\cos^2\theta} = \cos\theta$.
2. $\dfrac{\tan^2\theta + 1}{\tan\theta\csc^2\theta} = \tan\theta$.
3. $(1-\sin^2\theta)(1+\cot^2\theta) = \cot^2\theta$.
4. $\tan^2\theta - \cot^2\theta = \sec^2\theta - \csc^2\theta$.
5. $\sin\theta\sec\theta\sqrt{\csc^2\theta - 1} = 1$.
6. $\sec^2\theta + \csc^2\theta = \sec^2\theta\csc^2\theta$.
7. $(1+\tan^2\theta)(1-\sin^2\theta) = 1$.
8. $(1+\tan\theta)^2 + (1-\tan\theta)^2 = 2\sec^2\theta$.
9. $\sec^2\theta\cot^2\theta - 1 = \cot^2\theta$.
10. $\dfrac{1}{1-\sin\theta} + \dfrac{1}{1+\sin\theta} = 2\sec^2\theta$.
11. $\dfrac{1}{\sin^2\theta} - \dfrac{1}{\tan^2\theta} = 1$.
12. $(\tan\theta + \sec\theta)^2 = \dfrac{1+\sin\theta}{1-\sin\theta}$.

13. $\sin^4\theta - \cos^4\theta = 1 - 2\cos^2\theta$.

14. $\dfrac{\tan^2\theta + 1}{\tan^2\theta} = \csc^2\theta$.

15. $\sin^2\theta\tan\theta + \cos^2\theta\cot\theta + 2\sin\theta\cos\theta = \tan\theta + \cot\theta$.

16. Find the value of acute angle α in each of the following equations:
(a) $\cos 2\alpha = \sin 7\alpha$ (b) $\tan 3\alpha = \cot 2\alpha$ (c) $\sec\alpha = \csc 5\alpha$

17. Prove the identity $\cos(90° - \alpha)\tan(90° - \alpha) = \cos\alpha$.

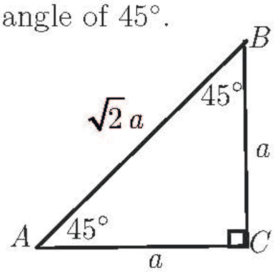
18. Prove the identity $\sin(90° - \alpha)\sec(90° - \alpha) = \cot\alpha$.

10.6 Value of the Trigonometric Ratios for Some Special Angles

Trigonometric Ratios for an Angle of 45°

In a 45°-45° right triangle by considering one with the two equal sides of length a units each, the hypotenuse is $\sqrt{2}a$.

We can now write the six trigonometric ratios for an angle of 45°.



$$\sin 45° = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45° = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45° = \frac{a}{a} = 1$$

The other three ratios are the reciprocals of these; thus $\csc 45° = \sqrt{2}$, $\sec 45° = \sqrt{2}$, $\cot 45° = 1$; or they may be read off from the figure.

Trigonometric Ratios for an Angle of 30° and 60°

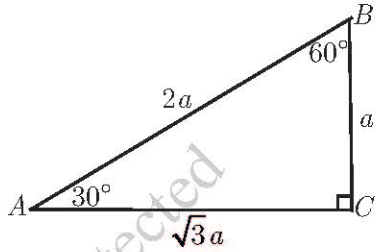
In a 30°-60° right triangle with the shorter leg of a units in length, the hypotenuse is $2a$ and the length of the other leg is $\sqrt{3}a$.

We can now write the six trigonometric ratios for an angle of 30° and 60°.

$$\sin 30^\circ = \frac{a}{2a} = \frac{1}{2}$$

$$\cos 30^\circ = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{a}{\sqrt{3}a} = \frac{\sqrt{3}}{3}$$



Again, $\sin 60^\circ = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$

$$\cos 60^\circ = \frac{a}{2a} = \frac{1}{2}$$

$$\tan 60^\circ = \frac{\sqrt{3}a}{a} = \sqrt{3}$$

The other three ratios may be read off from the figure.

Table summarizing the trigonometric ratios for special angles.

Table 10.1

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$

Example 11.

Find the values of $\csc^3 60^\circ$ and $\sec 30^\circ \sin 60^\circ \cos 45^\circ$.

Solution

$$\csc^3 60^\circ = (\csc 60^\circ)^3 = \left(\frac{2\sqrt{3}}{3}\right)^3 = \frac{2\sqrt{3}}{3} \times \frac{2\sqrt{3}}{3} \times \frac{2\sqrt{3}}{3} = \frac{8\sqrt{3}}{9}$$

$$\sec 30^\circ \sin 60^\circ \cos 45^\circ = \frac{2\sqrt{3}}{3} \times \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$$

Example 12.

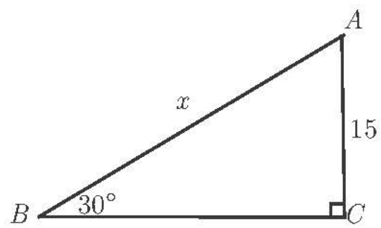
Find the value of $2\tan 45° + 2\sin^3 30° - 4\cos^4 30° + 3\tan^2 30°$.

Solution

$$\begin{aligned} & 2\tan 45° + 2\sin^3 30° - 4\cos^4 30° + 3\tan^2 30° \\ = \; & (2\times 1) + 2\left(\frac{1}{2}\right)^3 - 4\left(\frac{\sqrt{3}}{2}\right)^4 + 3\left(\frac{\sqrt{3}}{3}\right)^2 \\ = \; & 2 + \frac{1}{4} - \frac{9}{4} + 1 \\ = \; & 1 \end{aligned}$$

Example 13.

Find the side marked x from the given triangle.



Solution

To find x, we must choose a ratio that includes AB and AC. Thus

$$\begin{aligned} \csc B &= \frac{AB}{AC} \\ \csc 30° &= \frac{x}{15} \\ x &= 15\csc 30° \\ &= 15\times 2 = 30 \end{aligned}$$

Exercise 10.4

1. Draw a right triangle and find $\angle A$.

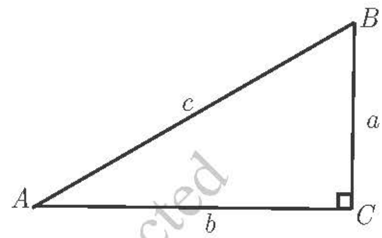
(a) $\sin A = \frac{1}{2}$ (b) $\cos A = \frac{\sqrt{3}}{2}$

(c) $\tan A = \sqrt{3}$ (d) $\cot A = 1$

(e) $\sec A = \sqrt{2}$ (f) $\csc A = 2$

For each of the right triangles ABC, find the indicated sides.

2. $\angle A = 30°$, $c = 30$, find a.
3. $\angle A = 60°$, $a = 15$, find b.
4. $\angle B = 45°$, $a = 16$, find c.
5. $\angle B = 30°$, $b = 8$, find c.



6. A ladder is placed along a wall such that it upper end is touching the top of the wall. The foot of the ladder is 5 ft away from the wall and the ladder is making an angle of 60° with the level of the ground. Find the height of the wall.

Find the numerical value of:

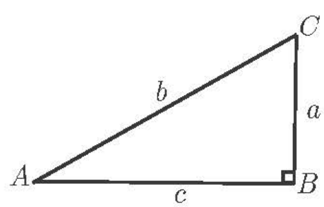
7. $\cot^3 45° + 4\sin^3 30°$
8. $\tan 60° \cot 30° + 4\sec^2 30°$
9. $\tan^2 45° + \sin 30° - \cos^2 30° + 2\tan^2 60°$
10. $\frac{1}{2}\sec^2 30° + \csc^2 45° - 2\tan^2 30°$

10.7 Solution of Right Triangles

Every triangle has six parts, namely, three sides and three angles. In the solution of right triangles there are really only two cases to be considered:

Case (1) To solve a right triangle when two sides are given.

Let $\triangle ABC$ be a right triangle with $\angle B = 90°$.
Suppose that any two sides are given.
Then the third side may be found from the equation $b^2 = a^2 + c^2$. Also $\sin A = \dfrac{a}{b}$, and $\angle C = 90° - \angle A$; whence $\angle A$ and $\angle C$ may be obtained.

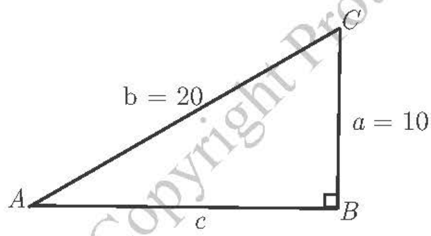


Case(2) To solve a right triangle one side and one acute angle are given.

Let $\triangle ABC$ be a right triangle with $\angle B = 90°$.
Suppose that one side c and one acute angle A are given.
Then $\angle C = 90° - \angle A$, $\frac{b}{c} = \sec A$, and $\frac{a}{c} = \tan A$;
whence $\angle C$, b and a may be obtained.

Example 14.

Solve the triangle ABC with $\angle B = 90°$, $a = 10$, $b = 20$.
Solution



Here $c^2 = b^2 - a^2 = 400 - 100 = 300$.

$$c = 10\sqrt{3}$$

$$\sin A = \frac{a}{b} = \frac{10}{20} = \frac{1}{2}$$

$$\therefore \angle A = 30°$$

$\angle C = 90° - \angle A = 90° - 30° = 60°$.

Alternative solution

$$\cos C = \frac{a}{b} = \frac{10}{20} = \frac{1}{2}$$

$$\therefore \angle C = 60°$$

$\angle A = 90° - \angle C = 90° - 60° = 30°$

$$\cos A = \frac{c}{b}$$

$$\cos 30° = \frac{c}{20}$$

$\therefore c = 20\cos 30° = 20 \times \frac{\sqrt{3}}{2} = 10\sqrt{3}$.

Example 15.

Solve the triangle ABC with $\angle B = 90°, \angle A = 30°,\ c = 6$.
Solution

Here $\angle C = 90° - \angle A = 90° - 30° = 60°$.

$$\frac{a}{c} = \tan A$$

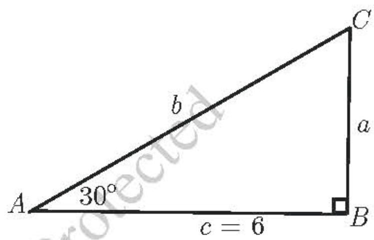
$$\frac{a}{6} = \tan 30°$$

$$\therefore a = 6\tan 30° = 6 \times \frac{\sqrt{3}}{3} = 2\sqrt{3}.$$

$$\frac{b}{c} = \sec A$$

$$\frac{b}{6} = \sec 30°$$

$$\therefore b = 6\sec 30° = 6 \times \frac{2}{\sqrt{3}} = 4\sqrt{3}.$$



Example 16.

In $\triangle ABC$ the angles A and C are equal to $30°$ and $120°$ respectively, and the side $AC = 20$ ft, find the length of the perpendicular from B upon AC produced.

Solution Draw BD perpendicular to AC produced.

Then $\theta = 180° - 120° = 60°$

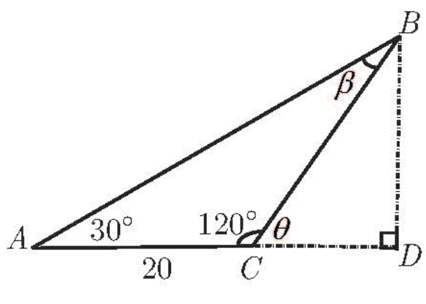
and $\beta = 180° - (30° + 120°) = 30°$.

$\therefore \angle A = \beta = 30°$ and $AC = BC = 20$.

In right $\triangle CDB$, $\dfrac{BD}{BC} = \sin\theta$

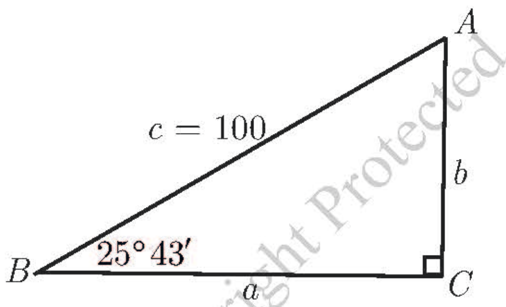
$$\frac{BD}{20} = \sin 60°$$

$$BD = 20\sin 60° = 20 \times \frac{\sqrt{3}}{2} = 10\sqrt{3} \text{ ft}.$$



Example 17.

Solve the triangle ABC with $\angle C = 90°, \angle B = 25°43'$ and $C = 100$. Using as much of the information below as necessary.
$[\cos 25°43' = 0.9010, \tan 25°43' = 0.4817]$.



Solution

$\angle A = 90° - \angle B = 90° - 25°43' = 64°17'$

$$\begin{aligned} \frac{a}{c} &= \cos B \\ a &= c\cos B = 100 \times \cos 25°43' \\ &= 100 \times 0.9010 \quad = 90.10 \\ \frac{b}{a} &= \tan B \\ b &= a\tan B = 90.10 \times \tan 25°43' \\ &= 90.12 \times 0.4817 \quad = 43.41. \end{aligned}$$

Exercise 10.5

1. Solve the triangles:
   (a) $\angle A = 90°, \quad a = 4, \quad c = 2\sqrt{3}.$
   (b) $\angle B = 90°, \quad c = 6, \quad b = 12.$
   (c) $\angle C = 90°, \quad \angle A = 30°, \quad a = 6\sqrt{3}.$
   (d) $\angle A = 30°, \quad \angle B = 60°, \quad b = 10\sqrt{3}.$

2. Given $\triangle ABC$ with $\angle A = 30°$, $\angle B = 135°$ and $AB = 100$, find the length of the perpendicular from C to AB produced.

3. If BD is perpendicular to the base AC of a triangle ABC, find a and c, given $\angle A = 30°$, $\angle C = 45°, BD = 10$.

4. In the triangle ABC, the angles B and C are equal to $45°$ and $120°$ respectively; if $a = 40$, find the length of the perpendicular from A on BC produced.

5. Solve the isosceles triangle. (Draw the perpendicular from the vertex to the base.)

   (a) $b = c = 12, \quad \angle B = 30°$.

   (b) $a = b = 9, \quad \angle A = 60°$.

   (c) $b = c = 10, \quad \angle A = 120°$.

6. In $\triangle ABC$, $\angle A = 60°$, $AC = 12$ and $AB = 20$. Find the length of the perpendicular drawn from C to AB. Also find the area of the triangle ABC.

7. Find the area of the triangle ABC, given $AB = 30$, $BC = 16$ and $\angle B = 30°$.

8. Solve the triangle ABC with $\angle B = 90°, \angle A = 36°$ and $c = 100$. Using as much of the information below as necessary.

   $[\tan 36° = 0.7265, \sec 36° = 1.2361]$

9. Solve the triangle ABC with $\angle A = 90°, c = 37$ and $a = 100$. Using as much of the information below as necessary.

   $[\sin 21°43' = 0.3700, \cos 21°43' = 0.9290]$

10.8 Angle of Elevation and Angle of Depression

Let OA be a horizontal line in the same vertical plane as an object B. Let O and B be joined.

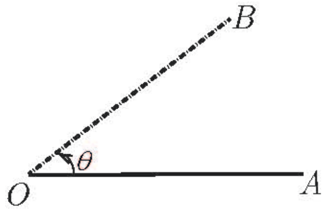


Figure 10.6(a)

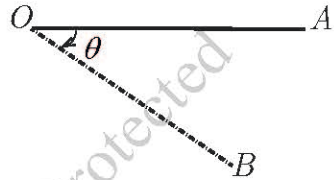


Figure 10.6(b)

In Fig. 10.6(a), where the object B is above the horizontal line OA, θ is called **the angle of elevation** of the object B as seen from the point O.

In Fig. 10.6(b), where the object B is below the horizontal line OA, θ is called **the angle of depression** of the object B as seen from the point O.

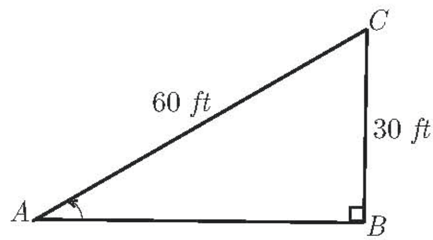
Example 18.

A kite is at the end of a 60 ft string that is taut. It is 30 ft above the ground. What is the angle of elevation of the kite?

Solution

$$\sin A = \frac{30}{60} = \frac{1}{2}$$

$$\angle A = 30^\circ$$

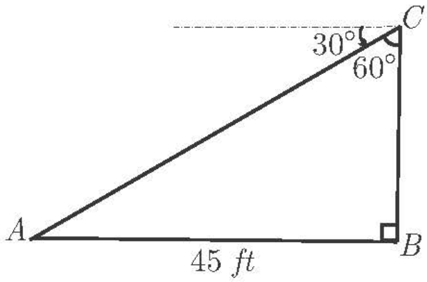


Therefore, the angle of elevation of the kite is 30°.

Example 19.

From the top of a house the angle of depression to a point on the ground is 30°. The point is 45 ft from the base of the building. How high is the building?

Solution



The angle of depression is not within the triangle. However, the complement of the angle 60°, is the measure of $\angle B$ inside the triangle.

Thus $\cot 60° = \frac{BC}{AC}$

$$BC = AC \cot 60°$$

$$= 45 \times \frac{1}{\sqrt{3}}$$

$$= 15\sqrt{3} \text{ ft}$$

Exercise 10.6

1. A mountain railway runs for 400 yards at a uniform slope of 30° with the horizontal. What is the horizontal distance between its two ends?

2. A vertical mast is secured from its top by straight cables 600 ft long fixed into the ground. Each cable makes angle of 60° with the ground. What is the height of the mast?

3. A kite flying at a height of 60 yards is attached to a string inclined at 45° to the horizontal. What is the length of the string? Assume there is no slack in the string.

4. An observer, 6 ft tall, is 20 yards away from a tower 22 yards high. Determine the angle of elevation from his eye to the top of the tower.

5. Two observers are on the opposite sides of a tower. They measure the angles of elevation to the top of the tower as 30° and 45° respectively. If the height of the tower is 40 yards, find the distance between them.

6. Find the angle of elevation of the sun when the shadow of a pole 12 ft high is $4\sqrt{3}$ ft long.

7. Two masts are 60 ft and 40 ft high, and the line joining their tops makes an angle of 30° with the horizon, find the distance between them.

8. From the foot of a tower, the angle of elevation to the top of a 60 ft column is 60°. The angle of elevation from the top of the tower to the top of the column is 30°. Find the height of the tower.